Module 3:Digital Electronics
Boolean Algebra

Quote of the day

“Most people say that it is the intellect which makes a great scientist. They are wrong: it is character”.

— Albert Einstein

Postulates that describe Boolean Algebra

P1: The Operations (+) and (·) are closed: i.e. $x, y \in B$

a) $x + y \in B$

b) $x \cdot y \in B$

P2:Identity Law

a) $0 + x = x + 0 = x$ for every $x \in B$

b) $x \cdot 1 = 1 \cdot x = x$ for every $x \in B$

P3:Commutative Law, for all $x, y \in B$

a) $x + y = y + x$

b) $x \cdot y = y \cdot x$

Postulates that describe Boolean Algebra

P4: Distributive Law: for all x, y, z $\in$ B

a) $x + (y \cdot z) = (x + y) \cdot (x + z)$

b) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

P5: Compliment Law for every element x in B there exist an element $\bar{x}$ such that

a) $x + \bar{x} = 1$

b) $x \cdot \bar{x} = 0$

P6: There exist at least two elements x, y $\in$ B such that $x \neq y$.

The algebraic expression written using variables is referred as Boolean expression.

Principal of Duality

- The **dual** of an algebraic expression is obtained by interchanging **+** and · and interchanging **0's** and **1's**.
- The Boolean identities appear in **dual** pairs.
- When there is only one identity on a line the identity is self-dual, i. e., the dual expression = the original expression.
- Sometimes, the dot symbol '•' (AND operator) is not written when the meaning is clear.

Summary of Postulates that describe Boolean Algebra

Postulate	Part (a)	Dual: Part (b)
P2:Identity Law	$0 + x = x + 0 = x$	$x \cdot 1 = 1 \cdot x = x$
P3:Commutative Law	$x + y = y + x$	$x \cdot y = y \cdot x$
P4: Distributive Law	$x + yz = (x + y)(x + z)$	$x(y + z) = xy + xz$
P5:Compliment Law	$x + \bar{x} = 1$	$x \cdot \bar{x} = 0$

Boolean Theorems

- Theorem 1:

a) $x + x = x$

Proof:

$x+x = (x+x)1$ by P2b

$=(x+x)\cdot(x+\bar{x})$ by P5a

$=x+x\cdot\bar{x}$ by P4a

$=x+0$ by P5b

$=x$ by P2a

Dual of Theorem 1:

b) $x \cdot x = x$

Proof:

$x\cdot x = (x\cdot x)+0$ by P2a

$=(x\cdot x)+(x\cdot\bar{x})$ by P5b

$=x\cdot(x+\bar{x})$ by P4b

$=x\cdot 1$ by P5a

$=x$ by P2b

Boolean Theorems

- Theorem 2:

  a) $x + 1 = 1$

  Proof:

  $x+1 = (x+1) \cdot 1$ by P2b

  $=(x+1)\cdot(x+\bar{x})$ by P5a

  $=x+1\cdot\bar{x}$ by P4a

  $=x+\bar{x}$ by P2b

  $=1$ by P5a

Dual of Theorem 2:

b) $x \cdot 0 = 0$

Proof:

$x\cdot 0 = (x\cdot 0)+0$ by P2a

$=(x\cdot 0)+(x\cdot\bar{x})$ by P5b

$=x\cdot(0+\bar{x})$ by P4b

$=x\cdot\bar{x}$ by P2a

$=0$ by P5b

Boolean Theorems

- Theorem 3:Involution law

Proof: Let $\bar{x}$ be the compliment of x and ($\bar{\bar{x}}$) be the compliment of $\bar{x}$.

Then by postulate P5 $x + \bar{x} = 1$, $x \cdot \bar{x} = 0$, $\bar{x} + (\bar{\bar{x}}) = 1$ and $\bar{x} \cdot (\bar{\bar{x}}) = 0$.

$$\overline{(\bar{x})} = \overline{(\bar{x})} + 0 \qquad By \quad P2a$$

$$= \overline{(\bar{x})} + x\bar{x} \qquad By \quad P5b$$

$$= \left(\overline{(\bar{x})} + x\right) \cdot \left(\overline{(\bar{x})} + \bar{x}\right) \quad By \quad P4a$$

$$= \left(\overline{(\bar{x})} + x\right) \bullet (1) \qquad By \quad P5a$$

Boolean Theorems

- Theorem 3:Involution law

Proof contd: $\overline{(\overline{x})} = \left(\overline{(\overline{x})} + x\right) \bullet \left(\overline{x} + x\right) \quad By \quad P5a$

$$= x + \left(\overline{(\overline{x})} \cdot \overline{x}\right) \quad By \quad P4a$$

$$= x + 0 \quad By \quad P5b$$

$$= x \quad By \quad P2a$$

Boolean Theorems

- Theorem 4:Associative Law (Proof By Perfect Induction)

  a) $x + (y + z) = (x + y) + z$

Associative Law						
X	Y	Z	X+Y	Y+Z	(X + Y) + Z	X + (Y + Z)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Boolean Theorems

- Theorem 4:Associative Law (Proof By Perfect Induction)

  b) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Associative Law						
X	Y	Z	X·Y	Y·Z	(X · Y) · Z	X· (Y · Z)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Boolean Theorems

- Theorem 5:Demorgans Theorem By Perfect Induction

  a)$(\overline{x + y}) = (\bar{x}+\bar{y})$

  b) $(\overline{x \cdot y}) = \bar{x} \cdot \bar{y}$

Demorgan’s Theorem									
X	Y	$\bar{X}$	$\bar{Y}$	$X \cdot Y$	$X+Y$	$\bar{X} \cdot \bar{Y}$	$\overline{x+y}$	$\bar{X}+\bar{Y}$	$\overline{x \cdot y}$
0	0	1	1	0	0	1	1	1	1
0	1	1	0	0	1	0	0	1	1
1	0	0	1	0	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0

Boolean Theorems

- Theorem 6:Absorption Law

  a) $x + x \cdot y = x$

Proof:

$x+(x \cdot y)= (x\cdot 1) +(x\cdot y)$.... ...... by P2b

$=x \cdot (1+y)$ by P4b

$=x \cdot 1$ by T2a

$=x$ by P2a

Dual of Theorem 2:

b) $x \cdot (x + y) = x$

Proof:

$x\cdot(x+y)= (x+0) \cdot (x+y)$.... ...... by P2a

$=x+0\cdot y$ by P4a

$=x+0$ by T2b

$=x$ by P2a

Summery of the Boolean Laws and theorems

P2:Identity Law	$0 + x = x + 0 = x$ $x \cdot 1 = 1 \cdot x = x$
P3:Commutative Law	$x + y = y + x$ $x \cdot y = y \cdot x$
P4: Distributive Law	$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$
P5:Compliment Law	$x + \bar{x} = 1$ $x \cdot \bar{x} = 0$
Idempotent law Theorem 1	$x + x = x$ $x \cdot x = x$
	$x + 1 = 1$ $x \cdot 0 = 0$
Involution Law	$\overline{(\bar{x})} = x$
Associative Law	$x + (y + z) = (x + y) + z$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Demorgans Theorem	$\overline{(x + y)} = \bar{x} \cdot \bar{y}$ $\overline{(x \cdot y)} = \bar{x} + \bar{y}$
Absorption law	$x + x \cdot y = x$ $x \cdot (x + y) = x$
a) $x + \bar{x} \cdot y = x + y$, b) $x \cdot (\bar{x} + y) = x \cdot y$	

Simplify $Y = AB + A\bar{B}$

- $Y = AB + A\bar{B}$ **By Distributive Theorem**
- $Y = A(B + \bar{B})$ **By Compliment law,**
- $Y = A.1$ **By Identity law,**
- $Y = A$

Simplify $Y = (A+B)\cdot(A+\bar{B})$

Simplify $Y = B(DC + D\bar{C}) + AB$

Simplify $Y = ABC + \bar{A}B + AB\bar{C}$

Reduce the following Boolean Expression

$$f = \left(\overline{A+B}\right)\cdot\left(\overline{A}+\overline{C}\right)\cdot\left(\overline{B}+C\right)$$

By Demorgan's Theorem

$$\therefore f = \left(\overline{A}\cdot\overline{B}\right)\cdot\left(\overline{A}+\overline{C}\right)\cdot\left(\overline{B}+C\right)$$

By Associative and commutative Law

$$\therefore f = \left(\overline{A}\right)\cdot\left(\overline{A}+\overline{C}\right)\cdot\overline{B}\cdot\left(\overline{B}+C\right)$$

By Absorption law

$$\therefore f = \left(\overline{A}\right)\cdot\left(\overline{B}\right)$$

$$\therefore f = \overline{A}\cdot\overline{B}$$

By Demorgan's Theorem

$$\therefore f = \left(\overline{A+B}\right)$$

Simplify Y= AB+ $\overline{AC}$+A$\bar{B}$C(AB+ C)

$$\therefore Y = AB + \overline{AC} + A\overline{B}CAB + A\overline{B}CC$$

By Distributive Theorem

$$\therefore Y = AB + \overline{AC} + AC(\overline{B} \cdot B) + A\overline{B}C$$

0

By Compliment law, Identity, Demorgan's

$$\therefore Y = AB + \overline{A} + \overline{C} + A\overline{B}C$$

By Distributive Theorem

$$\therefore Y = \overline{A} + \overline{C} + A(B + \overline{B}C)$$

By Absorption law

$$\therefore Y = \overline{A} + \overline{C} + A(B + C)$$

By Distributive Theorem

$$\therefore Y = \overline{A} + \overline{C} + AB + AC$$

By Associative Theorem

$$\therefore Y = \overline{A} + AB + \overline{C} + AC$$

By Absorption law

$$\therefore Y = \overline{A} + B + A + \overline{C}$$

By Associative law

$$\therefore Y = B + \overline{C} + \overline{A} + A$$

By Compliment law, Identity

$$\therefore Y = B + \overline{C} + 1 \quad \therefore Y = 1$$

Reduce the following Boolean Expression

$$f = ABC + A\overline{B}C + AB\overline{C} + \overline{A}BC$$

$$\therefore f = AC(B + \overline{B}) + AB\overline{C} + \overline{A}BC$$

By Distributive Theorem

$$\therefore f = AC + AB\overline{C} + \overline{A}BC$$

By Compliment law, Identity

$$\therefore f = A(C + B\overline{C}) + \overline{A}BC$$

By Distributive Theorem

$$\therefore f = A(C + B) + \overline{A}BC$$

By Absorption law

$$\therefore f = AC + AB + \overline{A}BC$$

By Distributive Theorem

$$\therefore f = AC + B(A + \overline{A}C)$$

By Distributive Theorem

$$\therefore f = AC + B(A + C)$$

By Absorption law

$$\therefore f = AC + AB + BC$$

By Distributive Theorem

Simplify the following Boolean Expression

$$ABC + \bar{A}\bar{B}C + \bar{A}BC + AB\bar{C} + \bar{A}\bar{B}\bar{C}$$

$$BC + A\bar{C} + AB + BCD$$

$$\left(\overline{\overline{CD} + A}\right) + A + CD + AB$$

Show that AB+ĀC+BC=AB+ĀC

Assignment

- With the help of switching circuit, Input/output waveforms and truth table explain the operation of a NOT Gate, AND Gate and OR gate.
- Explain the construction of OR and AND Gates using Diodes.
- Prove that a) $x + \bar{x} \cdot y = x + y$, b) $x \cdot (\bar{x} + y) = x \cdot y$