The z-Transform

Quote of the Day

Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories.

Laplace

Some Special Functions

First consider the delta function (or unit sample function):

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad \textbf{or} \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This allows an arbitrary sequence $x(n)$ or continuous-time function $f(t)$ to be expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$$

Convolution, Unit Step

$$x(n) = x(n) * \delta(n)$$

These are referred to as discrete-time convolution, and is denoted by:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Also consider the unit step function:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

z-Transform

- The z-transform is the most general concept for the transformation of discrete-time series.
- The Laplace transform is the more general concept for the transformation of continuous time processes.
- For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.
- The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Z Transforms

The one-sided z-transform of a function *x*(*n*): Also known as unilateral z-transform.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The two-sided z-transform of a function *x*(*n*): Bilateral z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Relationship to Fourier Transform

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$i.e.\, Z = re^{i\omega} \Longrightarrow X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n},\ or$$

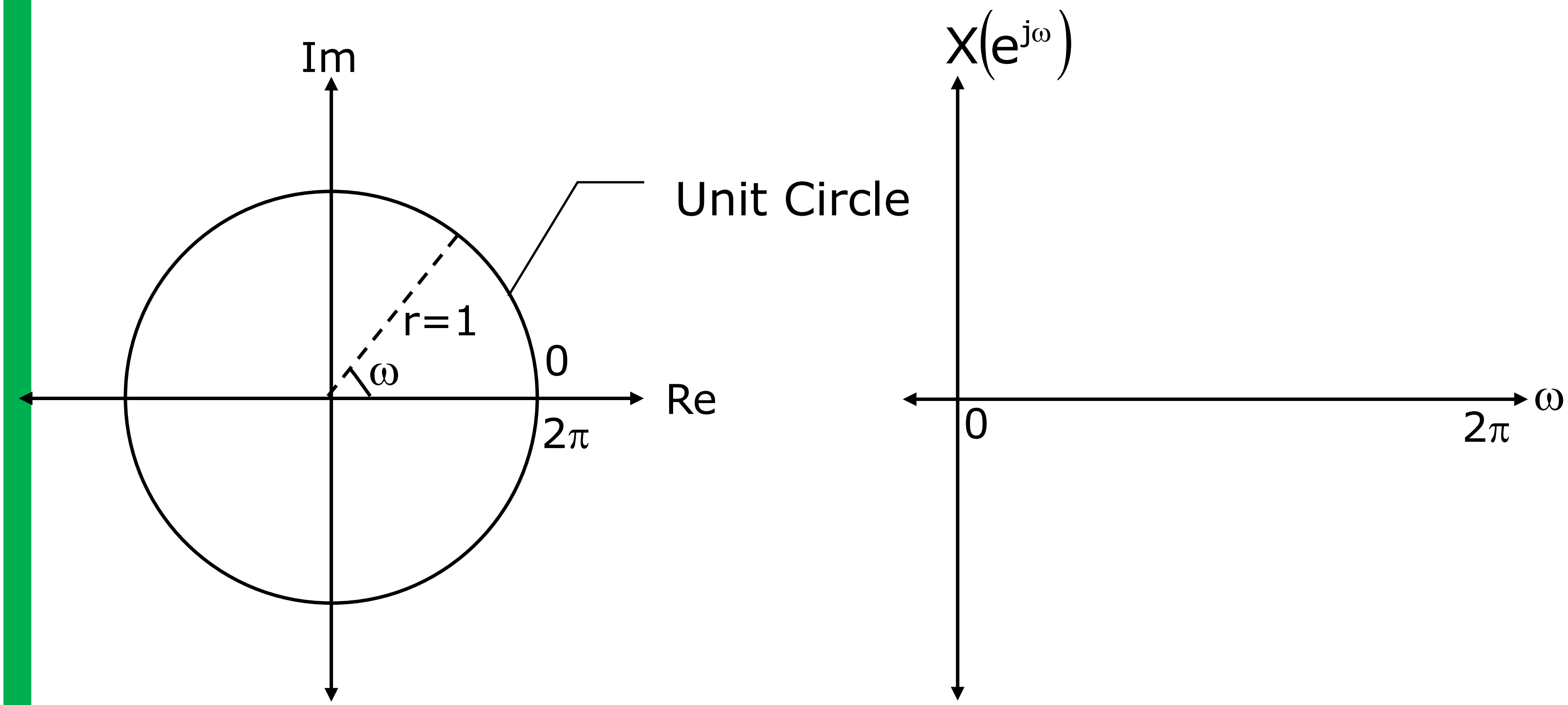
$$\therefore X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-i\omega n},\ and\ if\ \ r = 1,$$

$$\therefore X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}$$

which is the Fourier transform of $x(n)$.

The z-transform and the DTFT

- The z-transform is a function of the complex z variable
- It is convenient to describe on the complex z-plane
- If we plot $z=e^{j\omega}$ for $\omega=0$ to 2π we get the unit circle

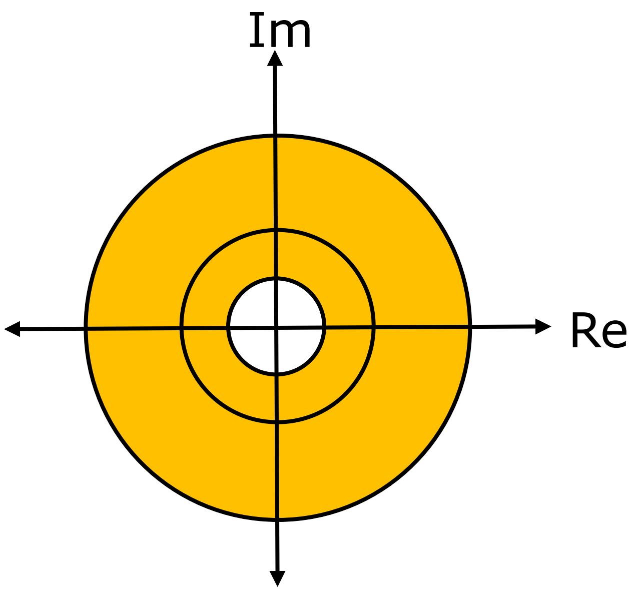


Region of Convergence(ROC)

The power series for the z-transform is called a Laurent series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \Rightarrow \therefore X(z) = ...x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2}.....$$

- The set of all values of z for which this series converges(attains the finite value) is known as its region of convergence.
- Each value of r represents a circle of radius r.



- The region of convergence is made of circles.
- Example: z-transform converges for values of 0.5<r<2
- Not all sequence have a z-transform
- Example: $x[n] = \cos(\omega_o n)$
  - Does not converge for any r
  - No ROC, No z-transform

Poles and Zeros

When $X(z)$ is a rational function, i.e., a ratio of polynomials in z, then:

1. The roots of the numerator polynomial are referred to as the zeros of $X(z)$, and
2. The roots of the denominator polynomial are referred to as the poles of $X(z)$.

Note that no poles of $X(z)$ can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Right-Sided Exponential Sequence Example

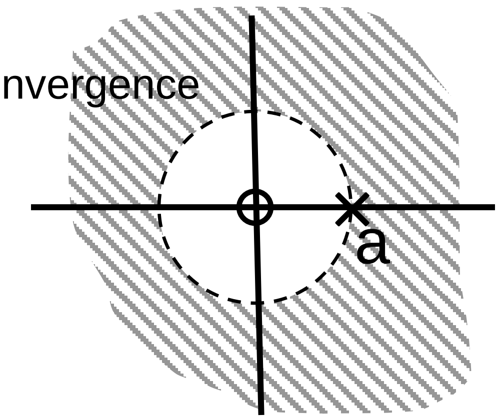
$$x[n]=a^n u[n] \quad \Rightarrow \quad X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty}\left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

$$\left|az^{-1}\right|<1 \Rightarrow |z|>|a|$$



- Inside the ROC series converges to

$$X(z)=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n = \frac{\left(az^{-1}\right)^0-\left(az^{-1}\right)^\infty}{1-az^{-1}} = \frac{1}{1-az^{-1}}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1}-a^{N_2+1}}{1-a}$$

$$\therefore X(z)=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Clearly, $X(z)$ has a zero at $z = 0$ and a pole at $z = a$.

Left-Sided Exponential Sequence Example

$$x[n] = -b^n u[-n-1] \;\Rightarrow\; X(z) = -\sum_{n=-\infty}^{\infty} b^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} \left(bz^{-1}\right)^n$$

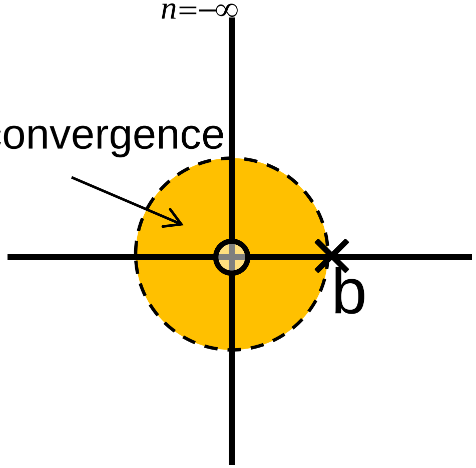
- We can alternatively write as

$$X(z) = \sum_{n=1}^{\infty} \left|bz^{-1}\right|^{-n} = \sum_{n=1}^{\infty} \left|b^{-1}z\right|^{n} < \infty$$

- Hence the ROC is defined as

$$\left|b^{-1}z\right| < 1 \Rightarrow |z| < |b|$$

Region of convergence



- Inside the ROC series converges to

$$X(z) = 1 - \sum_{n=0}^{\infty} \left(b^{-1}z\right)^n = 1 - \frac{\left(b^{-1}z\right)^0 - \left(b^{-1}z\right)^\infty}{1-b^{-1}z}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$\therefore X(z) = 1 - \frac{1}{1-b^{-1}z} = 1 - \frac{b}{b-z} = \frac{-z}{b-z}$$

$$\therefore X(z) = \frac{z}{z-b}$$

- Region Inside the circle of radius b is the ROC
- Left-sided sequence ROCs extend Inside a circle

Clearly, $X(z)$ has a zero at $z = 0$ and a pole at $z = b$.

Two-Sided Exponential Sequence Example

$$x[n]=\left(-\frac{1}{3}\right)^n u[n]-\left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n=\frac{\left(-\frac{1}{3}z^{-1}\right)^0-\left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}}=\frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n=\frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty}-\left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}}=\frac{-1}{1-\frac{1}{2}z^{-1}}$$

$$X(z)=\frac{z}{z+\frac{1}{3}}+\frac{z}{z-\frac{1}{2}}=\frac{z\left(2z-\frac{1}{6}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

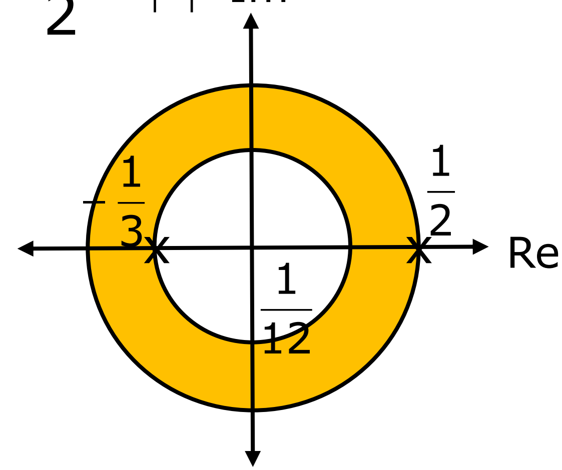
$$ROC: \frac{1}{3}<|z|<\frac{1}{2}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right|<1$$

$$\frac{1}{3}<|z|$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right|>1$$

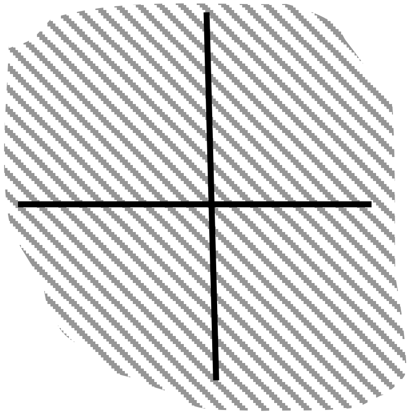
$$\frac{1}{2}>|z|$$



Clearly, $X(z)$ has a zero at z = 0,1/12 and a pole at z = ½,-1/3.

Finite Length Sequence

$$\text{If} \quad x[n]=\begin{cases}2^n & 0\le n\le 3\\ 0 & \text{otherwise}\end{cases}$$

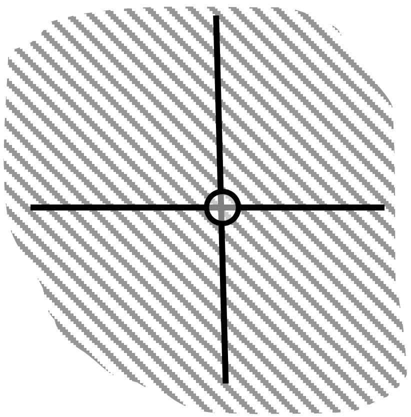


$$\therefore X(z)=\sum_{n=0}^{3}2^n z^{-n}=1+2z^{-1}+4z^{-2}+8z^{-3}$$

X(z) is the series of –ve powers of z. Therefore the ROC for finite-length right sided sequence is the entire z-plane except at z=∞

$$\text{If} \quad x[n]=\begin{cases}2^n & -3\le n\le 0\\ 0 & \text{otherwise}\end{cases}$$

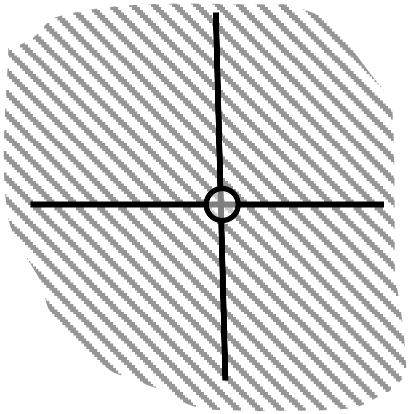
$$\therefore X(z)=\sum_{n=-3}^{0}2^n z^{-n}=2^{-3}z^{3}+2^{-2}z^{2}+2^{-1}z+1$$



X(z) is the series of +ve powers of z. Therefore the ROC for finite-length left sided sequence is the entire z-plane except at z=0

Finite Length Sequence contd..

$$x[n]=\begin{cases} a^n & -3\le n\le 0 \\ b^n & 0<n\le 3 \end{cases}$$



$$\therefore X(z)=\sum_{n=-3}^{0} a^n z^{-n}+\sum_{n=1}^{3} b^n z^{-n}$$

$$\therefore X(z)=a^{-3}z^{3}+a^{-2}z^{2}+a^{-1}z+1+bz^{-1}+b^{2}z^{-2}+b^{3}z^{-3}$$

X(z) is the series of –ve powers & +ve powers of z. Therefore the ROC for finite-length left sided sequence is the entire z-plane except at z=0 & z=∞

$$x[n]=\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z)=\sum_{n=0}^{N-1} a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}$$

- Region outside the circle of radius a is the ROC

Properties of The ROC of Z-Transform

- The ROC is a ring or disk centered at the origin.
- The ROC cannot contain any poles.
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including $z = \infty$.
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including $z=0$.
- The ROC of a two-sided sequence is a ring bounded by poles .
- The ROC must be a connected region.
- The ROC for finite-length sequence is the entire z-plane
  - except possibly $z=0$ and $z=\infty$.
- A z-transform does not uniquely determine a sequence without specifying the ROC.

What is the z-transform of delta function?

$$x(n) = \delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \Rightarrow \quad \therefore X(z) = \sum_{n=0}^{0} 1 \times z^{-n} = 1$$

$$\therefore \delta(n) \xleftrightarrow{ZT} 1$$ ROC is the entire z-plane

What is the z-transform of unit step function?

$$x(n) = u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \Rightarrow \quad \therefore X(z) = \sum_{n=0}^{\infty} 1 \times z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\therefore u(n) \xleftrightarrow{ZT} = \frac{z}{z-1}$$

- Region outside the circle of radius a is the ROC

  |Z|>1

Review

$$\delta(n) \xleftrightarrow{ZT} 1 \quad \text{Roc is entire z plane}$$

$$u(n) \xleftrightarrow{ZT} = \frac{z}{z-1} \quad \text{ROC is} |z| > 1$$

$$a^n u(n) \xleftrightarrow{ZT} = \frac{z}{z-a} \quad \text{ROC is} |z| > a$$

$$-b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-b} \quad \text{ROC is} |z| < b$$

$$a^n u(n) - b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-a} + \frac{z}{z-b} \quad \text{ROC is a} < |z| < b$$

$$\text{If } a > \text{b else ZT does not exit.}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$