The properties of z-Transform

Quote of the Day

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before.

But in poetry, it's the exact opposite.

Paul Dirac

Review

$$\delta(n) \xleftrightarrow{ZT} 1 \quad \text{Roc is entire z plane}$$

$$u(n) \xleftrightarrow{ZT} = \frac{z}{z-1} \quad \text{ROC is} |z| > |1|$$

$$a^n u(n) \xleftrightarrow{ZT} = \frac{z}{z-a} \quad \text{ROC is} |z| > |a|$$

$$-b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-b} \quad \text{ROC is} |z| < |b|$$

$$a^n u(n) - b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-a} + \frac{z}{z-b} \quad \text{ROC is } |a| < |z| < |b|$$

$$\text{If } a > b \text{ else ZT does not exit.}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

Z-Transform Properties: Linearity

- Notation

$$x[n] \overset{Z}{\longleftrightarrow} X(z) \qquad ROC = R_x$$

- Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z) \qquad ROC = R_{x_1} \cap R_{x_2}$$

  – Example: $x[n] = a^n u[n] - a^n u[n-N]$

    - Both sequences are right-sided
    - Both sequences have a pole z=a
    - Both have a ROC defined as |z|>|a|

- We did make use of this property already, where?
- Ans:

$$= \frac{z}{z-a} - \frac{a^N z^{-N} z}{z-a} = \frac{z^N - a^N}{z^{-N+1}(z-a)} = \frac{z^N - a^N}{z^{-(N-1)}(z-a)}$$

Determine the ZT of $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{3}\right)^n u(-n-1)$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{ZT} \frac{z}{z-\frac{1}{2}} \quad |z| < |\tfrac{1}{2}|$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{ZT} \frac{z}{z-\frac{1}{3}} \quad |z| < |\tfrac{1}{3}|$$

$$\therefore x(n) \xleftrightarrow{ZT} \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} \quad |z| < |\tfrac{1}{3}| \cap |z| < |\tfrac{1}{2}| \qquad |z| < |\tfrac{1}{3}|$$

Determine the ZT of $x[n] = \left(-\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[n]$

$$\therefore x(n) \xleftrightarrow{ZT} \frac{z}{z+\frac{1}{2}} - \frac{z}{z-\frac{1}{3}} \quad |z| > |\tfrac{1}{3}| \cap |z| > |\tfrac{1}{2}|$$

$$|z| > |\tfrac{1}{2}|$$

Find Z-transform of $cos(\omega_0 n)u(n)$

Here $x[n]=\cos(\omega_o n)u[n] \quad =\frac{1}{2}\left(e^{j\omega_o}\right)^n u[n]+\frac{1}{2}\left(e^{-j\omega_o}\right)^n u[n]$

Let $x_1(n)=\frac{1}{2}\left(e^{j\omega_o}\right)^n u[n] \,\&\, x_2(n)=\frac{1}{2}\left(e^{-j\omega_o}\right)^n u[n]$

By linearity property $X[z]=X_1[z]+X_2[z]$

$$X_1[z]=\frac{1}{2}\frac{z}{z-e^{j\omega_o}} \,\&\, X_2[z]=\frac{1}{2}\frac{z}{z-e^{-j\omega_o}} \quad |z|>|1|$$

$$\therefore X[z]=\frac{1}{2}\frac{z}{z-e^{j\omega_o}}+\frac{1}{2}\frac{z}{z-e^{-j\omega_o}} \quad |z|>|1|$$

$$\therefore X[z]=\frac{1}{2}\left[\frac{z\left(z-e^{-j\omega_o}\right)+z\left(z-e^{j\omega_o}\right)}{\left(z-e^{j\omega_o}\right)\left(z-e^{-j\omega_o}\right)}\right]=\frac{1}{2}\left[\frac{z^2-z\left(e^{-j\omega_o}\right)+z^2-z\left(e^{j\omega_o}\right)}{z^2-z\left(e^{j\omega_o}\right)-z\left(e^{-j\omega_o}\right)+1}\right]$$

$$\therefore X[z]=\frac{1}{2}\left[\frac{2z^2-z\left(e^{-j\omega_o}+e^{j\omega_o}\right)}{z^2-z\left(e^{j\omega_o}+e^{-j\omega_o}\right)+1}\right]=\left[\frac{z^2-z\cos(\omega_o)}{z^2-2z\cos(\omega_o)+1}\right]$$

$$\therefore \cos(\omega_o n)\xleftrightarrow{ZT}\left[\frac{z^2-z\cos(\omega_o)}{z^2-2z\cos(\omega_o)+1}\right] \mathrm{ROC}=|z|>|1|$$

Similarly find Z-transform of $sin(\omega_0 n)u(n)$

Here $x[n]=\sin(\omega_o n)u[n] = \frac{1}{2j}\left(e^{j\omega_o}\right)^n u[n] - \frac{1}{2j}\left(e^{-j\omega_o}\right)^n u[n]$

Let $x_1(n)=\frac{1}{2j}\left(e^{j\omega_o}\right)^n u[n] \,\&\, x_2(n)=\frac{1}{2j}\left(e^{-j\omega_o}\right)^n u[n]$

By linearity property $X[z]=X_1[z]-X_2[z]$

$$X_1[z]=\frac{1}{2j}\frac{z}{z-e^{j\omega_o}} \,\&\, X_2[z]=\frac{1}{2j}\frac{z}{z-e^{-j\omega_o}} \quad |z|>|1|$$

$$\therefore X[z]=\frac{1}{2j}\frac{z}{z-e^{j\omega_o}}-\frac{1}{2j}\frac{z}{z-e^{-j\omega_o}} \quad |z|>|1|$$

$$\therefore X[z]=\frac{1}{2j}\left[\frac{z\left(z-e^{-j\omega_o}\right)-z\left(z-e^{j\omega_o}\right)}{\left(z-e^{j\omega_o}\right)\left(z-e^{-j\omega_o}\right)}\right]=\frac{1}{2j}\left[\frac{z^2-z\left(e^{-j\omega_o}\right)-z^2+z\left(e^{j\omega_o}\right)}{z^2-z\left(e^{j\omega_o}\right)-z\left(e^{-j\omega_o}\right)+1}\right]$$

$$\therefore X[z]=\frac{1}{2j}\left[\frac{z\left(e^{j\omega_o}-e^{-j\omega_o}\right)}{z^2-z\left(e^{j\omega_o}+e^{-j\omega_o}\right)+1}\right]=\left[\frac{z\sin(\omega_o)}{z^2-2z\cos(\omega_o)+1}\right]$$

$$\therefore \sin(\omega_o n)\xleftrightarrow{ZT}\left[\frac{z\sin(\omega_o)}{z^2-2z\cos(\omega_o)+1}\right] \text{ROC}=|z|>|1|$$

Z-Transform Properties: Time Shifting

$$x[n-n_o] \stackrel{Z}{\longleftrightarrow} z^{-n_o}X(z) \qquad ROC = R_x$$

- Here n_o is an integer
  - If positive the sequence is shifted right
  - If negative the sequence is shifted left
- The ROC can change the new term
  - Add or remove poles at z=0 or z=∞
- Example

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

$$\mathcal{Z}[x(n-n_0)] = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n}$$

$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-(n-n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m)z^{-m}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m)z^{-m}$$

$$= z^{-n_0} X(z)$$

$$\therefore \left(\frac{1}{4}\right)^n u(n) \stackrel{ZT}{\longleftrightarrow} = \frac{z}{z-\frac{1}{4}} \quad \text{ROC is} |z| > \left|\frac{1}{4}\right| \quad \therefore X(z) = z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) = \frac{1}{z-\frac{1}{4}}$$

Find the z transform of $2^n u(n+2)$

We have to use the time shift property

$$x[n+n_o] \overset{Z}{\longleftrightarrow} z^{n_o} X[z] \qquad ROC = R_x$$

We know

$$2^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z-2} \qquad ROC = |z| > |2|$$

Using time shift property

$$2^{n+2} u(n+2) \overset{Z}{\longleftrightarrow} z^2 \frac{z}{z-2}$$

$$ROC = |z| > |2| \text{except at } z = \infty$$

Multiplying both sides by 2^{-2}

$$2^{-2} \times 2^{n+2} u(n+2) \overset{Z}{\longleftrightarrow} 2^{-2} \frac{z^3}{z-2}$$

$$2^n u(n+2) \overset{Z}{\longleftrightarrow} \frac{z^3}{4(z-2)}$$

$$ROC = |z| > |2| \text{except at } z = \infty$$

Using formula

$$X[z] = \sum_{n=-2}^{\infty} 2^n z^{-n}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$\therefore X[z] = \frac{2^{-2} z^2}{1-2z^{-1}} \qquad ROC = |z| > |2|$$

$$\therefore X[z] = \frac{z^3}{4(z-2)}$$

Z-Transform Properties: Multiplication by Exponential

ESTD:2014-15

$$r^n x[n] \xleftrightarrow{ZT} X(z/r) \qquad ROC = |r| R_x$$

$$Z\left[r^n x(n)\right] = \sum_{n=-\infty}^{\infty} r^n x(n) z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x(n)(r^{-1}z)^{-n}$$
$$= X(r^{-1}z)$$

- ROC is scaled by |r|
- All pole/zero locations are scaled
- If r is a positive real number: z-plane shrinks or expands
- If r is a complex number with unit magnitude it rotates

Example:

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{r^{-2}z^2 - r^{-1}z\cos(\omega_o)}{r^{-2}z^2 - 2r^{-1}z\cos(\omega_o) + 1} \qquad ROC = |z| > |r|$$

$$\therefore r^n \cos(\omega_o n) \xleftrightarrow{ZT} \left[\frac{z^2 - rz\cos(\omega_o)}{z^2 - 2rz\cos(\omega_o) + r^2}\right] \mathrm{ROC} = |z| > |r|$$

Similarly:

$$\therefore r^n \sin(\omega_o n) \xleftrightarrow{ZT} \left[\frac{r^{-1}z\sin(\omega_o)}{r^{-2}z^2 - 2r^{-1}z\cos(\omega_o) + 1}\right] \mathrm{ROC} = |z| > |r|$$

$$\therefore r^n \sin(\omega_o n) \xleftrightarrow{ZT} \left[\frac{rz\sin(\omega_o)}{z^2 - 2rz\cos(\omega_o) + r^2}\right] \mathrm{ROC} = |z| > |r|$$

Z-Transform Properties: Time Reversal

$$x[-n] \xleftrightarrow{Z} X(1/z) \qquad ROC = \frac{1}{R_x}$$

- ROC is inverted
- Example: $x[n] = a^{-n}u[-n]$

$$a^n u[n] \xleftrightarrow{ZT} \frac{z}{z-a} \quad |z| > |a|$$

$$\therefore X(z) = \frac{z^{-1}}{z^{-1} - a} = \frac{1}{1-az} \qquad |z| < |a^{-1}|$$

$$x[n] = 2^n u[-n] \qquad \therefore x[n] = \left(\frac{1}{2}\right)^{-n} u[-n]$$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{ZT} \frac{z}{z-\frac{1}{2}} \quad |z| > \left|\tfrac{1}{2}\right| \quad \therefore X(z) = \frac{z^{-1}}{z^{-1} - \frac{1}{2}} = \frac{1}{1-\frac{1}{2}z} = \frac{2}{2-z} \qquad |z| < \left|\left(\tfrac{1}{2}\right)^{-1}\right| \quad i.e.\ |z| < |2|$$

Z-Transform Properties: Time Reversal

$$x[n] = u[-n]$$

$$u[n] \xleftrightarrow{ZT} \frac{z}{z-1} \quad |z| > |1|$$

$$\therefore X(z) = \frac{z^{-1}}{z^{-1} - 1} = \frac{1}{1-z} \qquad |z| < |1|$$

Z-Transform Properties: Differentiation in z-domain

$$nx[n] \overset{ZT}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad ROC = R_x$$

$$\mathcal{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating both sides w.r.t. z

$$\frac{d}{dz}X[z] = \sum_{n=-\infty}^{\infty} x(n)\left(-nz^{-n-1}\right)$$

$$\therefore \frac{dX[z]}{dz} = -\sum_{n=-\infty}^{\infty} nx(n)z^{-n-1}$$

- Multiplying both sides by -z

$$\therefore -z\frac{dX[z]}{dz} = \sum_{n=-\infty}^{\infty} nx(n)z^{-n-1}z$$

$$\therefore -z\frac{dX[z]}{dz} = \sum_{n=-\infty}^{\infty} nx(n)z^{-n}$$

$$\therefore -z\frac{dX[z]}{dz} = \mathcal{Z}[nx(n)]$$

- Example: We want the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right) \qquad |z| > |a|$$

- Let's differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\left(\frac{1}{1+az^{-1}}\right)$$

- Term in bracket indicate z-transform of $(-a^n)$ & multiplication by z^{-1} indicate a time delay by 1.

$$\therefore nx[n] = a(-a)^{n-1}u[n-1]$$

$$\therefore x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

Z-Transform Properties: Differentiation in z-domain

- Find the z-transform of

$$x(n) = na^n u(n)$$

- We know that

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a} \quad \text{ROC} = |z| > |a|$$

- Now using property of differentiation

$$na^n u(n) \xleftrightarrow{ZT} -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$\therefore na^n u(n) \xleftrightarrow{ZT} \frac{az}{(z-a)^2} \quad \text{ROC} = |z| > |a|$$

- From this example we get

$$\therefore na^{n-1} u(n) \xleftrightarrow{ZT} \frac{z}{(z-a)^2} \quad \text{ROC} = |z| > |a|$$

- Find the z-transform of

$$x(n) = n2^{n-2} u(n)$$

- We can write

$$n2^n u(n) \xleftrightarrow{ZT} \frac{2z}{(z-2)^2} \quad \text{ROC} = |z| > |2|$$

- Multiplying both sides by 2^{-2}

$$n2^{n-2} u(n) \xleftrightarrow{ZT} \frac{2^{-2} \times 2 \times z}{(z-2)^2} \quad \text{ROC} = |z| > |2|$$

$$\therefore n2^{n-2} u(n) \xleftrightarrow{ZT} \frac{2^{-1} \times z}{(z-2)^2} \quad \text{ROC} = |z| > |2|$$

$$\therefore n2^{n-2} u(n) \xleftrightarrow{ZT} \frac{z}{2(z-2)^2} \quad \text{ROC} = |z| > |2|$$

Use properties to find z-transform of the following

- $x[n] = \sin\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)u[n-2]$.

$$\sin\left(\frac{\pi}{4}n\right)u[n] \xleftrightarrow{ZT} \frac{z}{\sqrt{2}\left(z^2 - \sqrt{2}z + 1\right)} \qquad \text{ROC is } |z| > |1|$$

$$\sin\left(\frac{\pi}{4}(n-2)\right)u[n-2] \xleftrightarrow{ZT} z^{-2}\frac{z}{\sqrt{2}\left(z^2 - \sqrt{2}z + 1\right)}$$

$$\sin\left(\frac{\pi}{4}(n-2)\right)u[n-2] \xleftrightarrow{ZT} \frac{z^{-1}}{\sqrt{2}\left(z^2 - \sqrt{2}z + 1\right)}$$

Use properties to find z-transform of the following

- $x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n]$.

$$\sin\left(\frac{\pi}{4}n\right)u[n] \xleftrightarrow{ZT} \frac{z}{\sqrt{2}\left(z^2 - \sqrt{2}z + 1\right)} \quad \text{ROC is } |z| > |1|$$

$$\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n] \xleftrightarrow{ZT} \left(\frac{z}{\sqrt{2}\left(z^2 - \sqrt{2}z + 1\right)}\right)\Bigg|_{z = 2z} \quad \text{ROC is } |z| > \left|\frac{1}{2}\right|$$

$$\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n] \xleftrightarrow{ZT} \frac{2z}{\sqrt{2}\left(4z^2 - 2\sqrt{2}z + 1\right)}$$

$$\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n] \xleftrightarrow{ZT} \frac{\sqrt{2}z}{\left(4z^2 - 2\sqrt{2}z + 1\right)}$$

Use properties to find z-transform of the x[n]=n(n+1)a^nu[n]

- $x[n]=n(n+1)a^n u[n]$,

$$na^n u[n] \xleftrightarrow{ZT} \frac{az}{(z-a)^2} \quad \text{ROC is } |z|>|a|$$

$$n \bullet na^n u[n] \xleftrightarrow{ZT} -z\frac{d}{dz}\frac{az}{(z-a)^2} \quad \text{ROC is } |z|>|a|$$

$$n \bullet na^n u[n] \xleftrightarrow{ZT} -z\frac{a(z-a)^2-2az(z-a)}{(z-a)^4} = -z\frac{az^2-2a^2z+a^3-2az^2+2a^2z}{(z-a)^4}$$

$$n \bullet na^n u[n] \xleftrightarrow{ZT} = -z\frac{-az^2+a^3}{(z-a)^4} = \frac{az\left(z^2-a^2\right)}{(z-a)^4} = \frac{az(z+a)(z-a)}{(z-a)^4} = \frac{az(z+a)}{(z-a)^3}$$

$$n \bullet na^n u[n] + na^n u[n] \xleftrightarrow{ZT} \frac{az(z+a)}{(z-a)^3} + \frac{az}{(z-a)^2} = \frac{az(z+a)+az(z-a)}{(z-a)^3}$$

$$n(n+1)a^n u[n] \xleftrightarrow{ZT} \frac{2az^2}{(z-a)^2} \qquad \text{ROC is } |z|>|a|$$

Use properties to find z-transform of the following

- x[n]=n(n-1)a^nu[n] .

$$na^n u[n] \xleftrightarrow{ZT} \frac{az}{(z-a)^2} \qquad \text{ROC is} |z| > |a|$$

$$n \bullet na^n u[n] \xleftrightarrow{ZT} \frac{az(z+a)}{(z-a)^3} \qquad \text{ROC is} |z| > |a|$$

$$n \bullet na^n u[n] - na^n u[n] \xleftrightarrow{ZT} \frac{az(z+a)}{(z-a)^3} - \frac{az}{(z-a)^2} = \frac{az(z+a) - az(z-a)}{(z-a)^3}$$

$$\therefore n(n-1)a^n u[n] \xleftrightarrow{ZT} \frac{2a^2 z}{(z-a)^2} \qquad \text{ROC is} |z| > |a|$$

$$\therefore n(n-1)\frac{a^{n-2}}{2} u[n] \xleftrightarrow{ZT} \frac{z}{(z-a)^2} \qquad \text{ROC is} |z| > |a|$$

If $x[n] \leftrightarrow z^2/(z^2-16)$ with ROC $|z|>4$ then obtain Y(Z) for y[n] given below

1. $y[n]=(1/2)^n x[n]$

$$\therefore \left(\frac{1}{2}\right)^n x[n] \xleftrightarrow{ZT} \frac{4z^2}{4z^2-16} = \frac{z^2}{z^2-4} \qquad \text{ROC is} |z| > |2|$$

2. $y[n]=nx[n]$

$$\therefore nx[n] \xleftrightarrow{ZT} -z\frac{(z^2-16)2z - z^2 \cdot 2z}{(z^2-16)^2} = -z\frac{2z^3-32z-2z^3}{(z^2-16)^2}$$

$$\therefore nx[n] \xleftrightarrow{ZT} \frac{32z^2}{(z^2-16)} \qquad \text{ROC is} |z| > |4|$$

3. $y[n]=x[n-2]$

$$\therefore x[n-2] \xleftrightarrow{ZT} z^{-2}\frac{z^2}{z^2-16} = \frac{1}{z^2-16} \qquad \text{ROC is} |z| > |4|$$

Z-Transform Properties: Convolution of two sequences

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z) \qquad ROC: R_{x1} \cap R_{x_2}$$

- Consider the convolution of $x_1(n)$ and $x_2(n)$ defined as

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

- The Z-transform of x (n) is defined as

$$X[z] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

- Interchanging the order of summation

$$X[z] = \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}$$

- Using time shift property

$$X[z] = \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)z^{-k}}_{X_1[z]} X_2[z] \quad \Rightarrow X[z] = X_1[z]X_2[z]$$

- We have to use this property along with the Inverse z-transform

Find the z-transform & specify the ROC

$$\left(\frac{2}{3}\right)^n u[n] * 2^n u[-n-3]$$

$$\therefore \left(\frac{2}{3}\right)^n u[n] \xleftrightarrow{ZT} \frac{z}{z-\frac{2}{3}} = \frac{3z}{3z-2} \quad ROC\ |z| > \left|\frac{2}{3}\right|$$

$$\therefore -(2)^n u[-n-1] \xleftrightarrow{ZT} \frac{z}{z-2} \quad ROC\ |z| < |2|$$

$$\therefore (2)^{n+2} u[-(n+2)-1] \xleftrightarrow{ZT} -z^2 \frac{z}{z-2} \quad ROC\ |z| < |2|$$

$$\therefore 2^n u[-(n+2)-1] \xleftrightarrow{ZT} -\frac{z^3}{4(z-2)} \quad ROC\ |z| < |2|$$

$$\therefore \left(\frac{2}{3}\right)^n u[n] * 2^n u[-n-3] \xleftrightarrow{ZT} \frac{3z}{(3z-2)} \cdot \frac{-z^3}{4(z-2)} \quad ROC\ \left|\frac{2}{3}\right| < |z| < |2|$$

$$\therefore \left(\frac{2}{3}\right)^n u[n] * 2^n u[-n-3] \xleftrightarrow{ZT} \frac{-3z^4}{4(3z-2)(z-2)} \quad ROC\ \left|\frac{2}{3}\right| < |z| < |2|$$

Use the properties to find ZT of $n^2\left(\frac{1}{3}\right)^n u[n-2]$

$$\left(\frac{1}{3}\right)^{n-2} u[n-2] \xleftrightarrow{ZT} \frac{z^{-1}}{z-\frac{1}{3}} \qquad \text{ROC is} |z| > \left|\frac{1}{3}\right|$$

$$\left(\frac{1}{3}\right)^{n} u[n-2] \xleftrightarrow{ZT} \frac{3z^{-1}}{9(3z-1)} = \frac{z^{-1}}{3(3z-1)} \qquad \text{ROC is} |z| > \left|\frac{1}{3}\right|$$

$$n\left(\frac{1}{3}\right)^{n} u[n-2] \xleftrightarrow{ZT} -z\frac{d}{dz}\frac{z^{-1}}{3(3z-1)} = -z\frac{-(3z-1)z^{-2}-3z^{-1}}{3(3z-1)^2}$$

$$n\left(\frac{1}{3}\right)^{n} u[n-2] \xleftrightarrow{ZT} -z\frac{-6z^{-1}+z^{-2}}{3(3z-1)^2} = \frac{6-z^{-1}}{3(3z-1)^2}$$

$$n^2\left(\frac{1}{3}\right)^{n} u[n-2] \xleftrightarrow{ZT} -z\frac{d}{dz}\frac{6-z^{-1}}{3(3z-1)^2}$$

$$\therefore X(z) = -z\frac{(3z-1)^2 z^{-2} - 6(6-z^{-1})(3z-1)}{3(3z-1)^4} = \frac{36z-9+z^{-1}}{3(3z-1)^3}$$

Complex conjugate property

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*) \quad ROC: R_x$$

- The Z-transform of x (n) is defined as

$$X[z] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Then Z-transform of x^* (n) can be defined as

$$Z\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$\therefore Z\{x^*[n]\} = \left(\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n} \right)^*$$

$$\therefore Z\{x^*[n]\} = X^*(Z^*)$$

Z-Transform Properties: Initial Value Theorem

$$x(0) = \lim_{z\to\infty} X(z)$$

- We know that

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

- Expanding the equation we get

$$X[z] = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

- Taking limits as z→∞

$$\lim_{z\to\infty} X[z] = \lim_{z\to\infty}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots\right)$$

$$\therefore \lim_{z\to\infty} X[z] = x(0)$$

Z-Transform Properties: Final Value Theorem

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} \left[(1 - z^{-1}) X(z) \right]$$

$$\mathcal{Z}[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$\mathcal{Z}[x(k-1)] = z^{-1} X(z) = \sum_{k=0}^{\infty} x(k-1) z^{-k}$$

Hence $$\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} = X(z) - z^{-1} X(z)$$

$$\lim_{z \to 1} \left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} \right] = \lim_{z \to 1} \left[X(z) - z^{-1} X(z) \right]$$

$$\sum_{k=0}^{\infty} [x(k) - x(k-1)] = [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \cdots = x(\infty) = \lim_{k \to \infty} x(k)$$

$$\therefore \lim_{k \to \infty} x(k) = \lim_{z \to 1} \left[(1 - z^{-1}) X(z) \right]$$

Z-Transform Properties

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$a^n x[n]$	$X(z/a)$	$\|a\| R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x[-n]$	$X(1/z)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
Initial-value theorem:		
$x[n] = 0, \quad n < 0$	$\lim_{z \to \infty} X(z) = x[0]$	