Inverse z-Transform

Quote of the Day

"Not everything that counts can be counted, and not everything that can be counted counts." (Sign hanging in Einstein's office at Princeton)

--**Albert Einstein**

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Review

$$\delta(n) \xleftrightarrow{ZT} 1 \quad \text{Roc is entire z plane} \quad \delta(n-k) \xleftrightarrow{ZT} z^{-k}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$u(n) \xleftrightarrow{ZT} = \frac{z}{z-1} \quad \text{ROC is} |z| > |1|$$

$$a^n u(n) \xleftrightarrow{ZT} = \frac{z}{z-a} \quad \text{ROC is} |z| > |a|$$

$$-b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-b} \quad \text{ROC is} |z| < |b|$$

$$a^n u(n) - b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-a} + \frac{z}{z-b} \quad \text{ROC is } |a| < |z| < |b|$$

If $a > b$ else ZT does not exit.

$$(-a)^n u(n) \xleftrightarrow{ZT} = \frac{z}{z+a} \quad \text{ROC is} |z| > |a|$$

$$\therefore \cos(\omega_o n) \xleftrightarrow{ZT} \left[\frac{z^2 - z\cos(\omega_o)}{z^2 - 2z\cos(\omega_o) + 1}\right] \text{ROC} = |z| > |1|$$

$$\therefore \sin(\omega_o n) \xleftrightarrow{ZT} \left[\frac{z\sin(\omega_o)}{z^2 - 2z\cos(\omega_o) + 1}\right] \text{ROC} = |z| > |1|$$

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Review

$$\boxed{\therefore r^n \cos(\omega_o n)u[n] \xleftrightarrow{ZT} \left[\frac{z^2 - rz\cos(\omega_o)}{z^2 - 2rz\cos(\omega_o) + r^2}\right] \mathrm{ROC} = |z| > |r|}$$

$$\boxed{\therefore r^n \sin(\omega_o n)u[n] \xleftrightarrow{ZT} \left[\frac{rz\sin(\omega_o)}{z^2 - 2rz\cos(\omega_o) + r^2}\right] \mathrm{ROC} = |z| > |r|}$$

$$\boxed{na^{n-1}u(n) \xleftrightarrow{ZT} = \frac{z}{(z-a)^2} \quad \mathrm{ROC\ is} |z| > |a|}$$

$$\boxed{\frac{n(n-1)}{2} a^{n-2}u(n) \xleftrightarrow{ZT} = \frac{z}{(z-a)^3} \quad \mathrm{ROC\ is} |z| > |a|}$$

$$\boxed{\frac{n(n-1)(n-2)}{2 \cdot 3} a^{n-3}u(n) \xleftrightarrow{ZT} = \frac{z}{(z-a)^4} \quad \mathrm{ROC\ is} |z| > |a|}$$

$$\boxed{-na^{n-1}u(n) \xleftrightarrow{ZT} = \frac{z}{(z-a)^2} \quad \mathrm{ROC\ is} |z| < |a|}$$

$$\boxed{-\frac{n(n-1)}{2} a^{n-2}u(n) \xleftrightarrow{ZT} = \frac{z}{(z-a)^3} \quad \mathrm{ROC\ is} |z| < |a|}$$

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Z-Transform Properties review

Property	Description
Linearity	$ax_1[n]+bx_2[n] \xleftrightarrow{Z} aX_1(z)+bX_2(z)$ $ROC = R_{x_1} \cap R_{x2}$
Time shift	$x[n-n_o] \xleftrightarrow{Z} z^{-n_o}X(z)$ $ROC = R_x$
Multiplication by Exponential	$r^n x[n] \xleftrightarrow{ZT} X(z/r)$ $ROC = \|r\|R_x$
Time Reversal	$x[-n] \xleftrightarrow{Z} X(1/z)$ $ROC = \frac{1}{R_x}$
Differentiation in z-domain	$nx[n] \xleftrightarrow{ZT} -z\frac{dX(z)}{dz}$ $ROC = R_x$
Convolution in z-domain	$x_1[n]*x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z)$ $ROC : R_{x1} \cap R_{x_2}$
Final Value Theorem:	$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$
Initial Value Theorem:	$x(0) = \lim_{z\to\infty} X(z)$

The Inverse Z-Transform

- Formal inverse z-transform is based on a Cauchy integral
- Most of the time inverse z-transform is found by
  - Inspection method
  - Partial fraction expansion
  - Power series expansion
- Inspection Method
  - Make use of known z-transform pairs such as

$$a^n u[n] \xleftrightarrow{Z} \frac{z}{z-a} \qquad |z| > |a|$$

  - Example: The inverse z-transform of

$$X(z) = \frac{z}{z - \frac{1}{2}} \qquad |z| > \frac{1}{2} \qquad \rightarrow \qquad x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse Z-Transform by Partial Fraction Expansion

- Assume that a given z-transform can be expressed as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Eliminate –ve powers of z by multiplying Num. and Den. by Z^N

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \dots\dots\dots + b_M z^M}{a_0 z^N + a_1 z^{N-1} + \dots\dots\dots + a_N}$$

- Then apply partial fractional expansion to get

$$\frac{X(z)}{z} = \sum_{k=1}^{N} \frac{A_k}{z - p_k}$$

- This expr. we get if M≤N. If M>N then $X(z) = \sum_{r=0}^{M-N} B_r z^r + \sum_{k=1}^{N} \frac{A_k z}{z - p_k}$
  - B_r is obtained by long division
- The expr. represents all first order distinct poles
- Each term can be inverse transformed by inspection

Inverse Z-Transform by Partial Fraction Expansion

- To determine the coefficients $A_1, A_2 \ldots\ldots A_N$ we seek an expression $$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots\ldots\ldots + \frac{A_N}{z-p_N}$$
- Each coefficients are given as $$A_k = (z-p_k)\times\left.\frac{X(z)}{z}\right|_{z=p_k}$$
- It is easier to understand with examples

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}} = \frac{z^2}{z^2-\frac{3}{2}z-\frac{1}{2}} \quad \Rightarrow \frac{X(z)}{z} = \frac{z}{(z-1)(z-\frac{1}{2})}$$

$$\therefore \frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-\frac{1}{2})} \Rightarrow \text{Now } A_1 = (z-1)\times\left.\frac{X(z)}{z}\right|_{z=1} \;\&\; A_2 = (z-\tfrac{1}{2})\times\left.\frac{X(z)}{z}\right|_{z=\frac{1}{2}}$$

$$\therefore A_1 = \left.\frac{z\times(z-1)}{(z-1)(z-\frac{1}{2})}\right|_{z=1} = \frac{1}{1-\frac{1}{2}} = \frac{1}{{}^1\!/_2} = 2$$

$$\& \therefore A_2 = \left.\frac{z\times(z-\frac{1}{2})}{(z-1)(z-\frac{1}{2})}\right|_{z={}^1\!/_2} = \frac{\frac{1}{2}}{\frac{1}{2}-1} = \frac{{}^1\!/_2}{-{}^1\!/_2} = -1$$

$$\boxed{\begin{aligned} &\therefore \frac{X(z)}{z} = \frac{2}{(z-1)} - \frac{1}{(z-\frac{1}{2})} \\ &\therefore X(z) = \frac{2z}{(z-1)} - \frac{z}{(z-\frac{1}{2})} \end{aligned}}$$

Inverse Z-Transform by Partial Fraction Expansion

- To determine the coefficients $A_1, A_2 \ldots\ldots A_N$ we seek an expression $\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots\ldots + \frac{A_N}{z-p_N}$
- Each coefficients are given as $A_k = (z-p_k) \times \left.\frac{X(z)}{z}\right|_{z=p_k}$
- It is easier to understand with examples

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}} = \frac{z^2}{z^2-\frac{3}{2}z-\frac{1}{2}} \qquad \Rightarrow \frac{X(z)}{z} = \frac{z}{(z-1)(z-\frac{1}{2})}$$

$$\therefore \frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-\frac{1}{2})} \Rightarrow \text{Now } A_1 = (z-1) \times \left.\frac{X(z)}{z}\right|_{z=1} \ \& \ A_2 = (z-\tfrac{1}{2}) \times \left.\frac{X(z)}{z}\right|_{z=\frac{1}{2}}$$

$$\because \frac{2z}{(z-1)} \xleftrightarrow{IZT} 2u(n) \ \& \ \frac{z}{(z-\frac{1}{2})} \xleftrightarrow{IZT} \left(\tfrac{1}{2}\right)^n u(n)$$

$$\therefore x(n) = 2u(n) - \left(\tfrac{1}{2}\right)^n u(n)$$

Inverse Z-Transform by Partial Fraction Expansion

- For multiple order poles we get the expression as

$$X(z)=\sum_{r=0}^{M-N}B_r z^r+\sum_{k=1,k\neq i}^{N}\frac{A_k z}{Z-p_k}+\sum_{m=1}^{s}\frac{C_m z}{(z-p_i)^m}$$

- Now to compute the coefficients C_m

$$c_m=\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m\times\frac{X(z)}{z}\bigg|_{z=p_i}$$

- Now consider Example 1:

$$X(z)=\frac{z^2}{(z-1)(z-\frac{1}{2})(z+1)^2}\qquad\Rightarrow\frac{X(z)}{z}=\frac{z}{(z-1)(z-\frac{1}{2})(z+1)^2}$$

$$\therefore\frac{X(z)}{z}=\frac{z}{(z-1)(z-\frac{1}{2})(z+1)^2}=\frac{A_1}{(z-1)}+\frac{A_2}{(z-\frac{1}{2})}+\frac{C_1}{(z+1)}+\frac{C_2}{(z+1)^2}$$

- Obtain coefficients A_1,A_2 as computed in previous example

$$\therefore A_1=\frac{z\times(z-1)}{(z-1)(z-\frac{1}{2})(z+1)^2}\bigg|_{z=1}=\frac{1}{(1-\frac{1}{2})(1+1)^2}=\frac{1}{(1/2)4}=\frac{1}{2}$$

Inverse Z-Transform by Partial Fraction Expansion

$$\& \therefore A_2 = \left.\frac{z\times(z-\frac{1}{2})}{(z-1)(z-\frac{1}{2})(z+1)^2}\right|_{z=\frac{1}{2}} = \frac{\frac{1}{2}}{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}+1\right)^2} = \frac{1/2}{\left(-1/2\right)\left(9/4\right)} = -\frac{4}{9}$$

- Now to compute the coefficients C_1 &C_2 from X(Z)/Z

$$c_m = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m \times \left.\frac{X(z)}{z}\right|_{z=p_i}$$

$$c_1 = \frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}(z+1)^2 \times \left.\frac{X(z)}{z}\right|_{z=-1} \qquad \therefore c_1 = (z+1)^2 \times \left.\frac{z}{(z-1)(z-\frac{1}{2})(z+1)^2}\right|_{z=-1}$$

$$\therefore c_1 = \frac{-1}{(-1-1)(-1-\frac{1}{2})} = \frac{-1}{(-2)\left(3/2\right)} = -\frac{1}{3} \qquad \& c_2 = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z+1)^2 \times \left.\frac{X(z)}{z}\right|_{z=-1}$$

$$\therefore c_2 = \frac{d}{dz}(z+1)^2 \times \left.\frac{X(z)}{z}\right|_{z=-1} \qquad \therefore c_2 = \frac{d}{dz}\left.\left(\frac{z}{(z-1)(z-\frac{1}{2})}\right)\right|_{z=-1} = \frac{d}{dz}\left.\left(\frac{z}{z^2-\frac{3}{2}z-\frac{1}{2}}\right)\right|_{z=-1}$$

$$\therefore c_2 = \left.\left(\frac{(z^2-\frac{3}{2}z-\frac{1}{2})-z\left(2z-\frac{3}{2}\right)}{(z^2-\frac{3}{2}z-\frac{1}{2})^2}\right)\right|_{z=-1} = \left(\frac{\left[(-1)^2-\frac{3}{2}(-1)-\frac{1}{2}\right]+\left(2(-1)-\frac{3}{2}\right)}{\left[(-1)^2+\frac{3}{2}-\frac{1}{2}\right]^2}\right) = -\frac{3}{4}$$

Inverse Z-Transform by Partial Fraction Expansion

$$\therefore \frac{X(z)}{z} = \frac{z}{(z-1)(z-\frac{1}{2})(z+1)^2} = \frac{{}^{1}\!/_{2}}{(z-1)} + \frac{{}^{-4}\!/_{9}}{(z-\frac{1}{2})} + \frac{{}^{-1}\!/_{3}}{(z+1)} + \frac{{}^{-3}\!/_{4}}{(z+1)^2}$$

$$\therefore X(z) = \frac{z^2}{(z-1)(z-\frac{1}{2})(z+1)^2} = \frac{z}{(z-1)} - \frac{{}^{2}\!/_{3}\, z}{(z-\frac{1}{2})} - \frac{{}^{1}\!/_{3}\, z}{(z+1)} - \frac{{}^{3}\!/_{4}\, z}{(z+1)^2}$$

- Now by method of inspection

$$\frac{{}^{1}\!/_{2}\, z}{(z-1)} \xleftrightarrow{IZT} \tfrac{1}{2}u(n),\quad \frac{{}^{4}\!/_{9}\, z}{(z-\frac{1}{2})} \xleftrightarrow{IZT} \tfrac{4}{9}\left(\tfrac{1}{2}\right)^n u(n) \quad and \quad \frac{{}^{1}\!/_{3}\, z}{(z+1)} \xleftrightarrow{IZT} \tfrac{1}{3}\left(-1\right)^n u(n)$$

- Now for repeated root $\frac{{}^{3}\!/_{4}\, z}{(z+1)^2}$ use the formula

- $\therefore na^{n-1}u(n) \xleftrightarrow{ZT} \frac{z}{(z-a)^2}$ $\quad \text{ROC} = |z| > |a|$ here a=-1

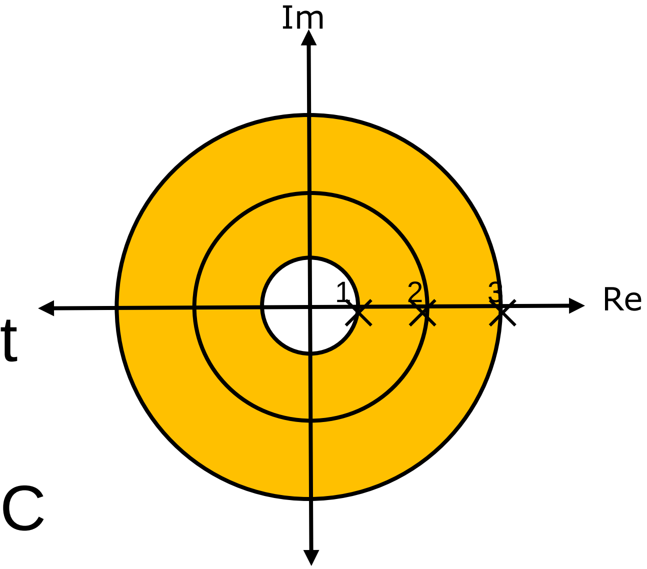
$$\therefore \frac{{}^{3}\!/_{4}\, z}{(z+1)^2} \xleftrightarrow{IZT} \tfrac{3}{4} n(-1)^{n-1}u(n)$$

$$\therefore x(n) = \tfrac{1}{2}u(n) - \tfrac{4}{9}\left(\tfrac{1}{2}\right)^n u(n) - \tfrac{1}{3}\left(-1\right)^n u(n) - \tfrac{3}{4} n(-1)^{n-1}u(n)$$

Inverse Z-Transform by Partial Fraction Expansion

Example 2.: Determine all possible signals x(n) associated with z-transform

$$X(z) = \frac{z(z^2 - 4z + 5)}{(z-1)(z-2)(z-3)}$$



- There are three poles for this X(z) at z=1,2,3.
- So there are four possibilities of ROC
  - i. |Z|<|1|
  - ii. |Z|>|3|
  - iii. |2|<|Z|<|3|
  - iv. |1|<|Z|<|2|
- First obtain coefficients A_1, A_2, A_3 of X(z)/z

$$\frac{X(z)}{z} = \frac{(z^2 - 4z + 5)}{(z-1)(z-2)(z-3)}$$

- Here M=2 and N=3 i.e. M<N so

Inverse Z-Transform by Partial Fraction Expansion

Example 2.: contd..

$$\frac{X(z)}{z}=\frac{(z^2-4z+5)}{(z-1)(z-2)(z-3)}=\frac{A_1}{(z-1)}+\frac{A_2}{(z-2)}+\frac{A_3}{(z-3)}$$

$$\therefore A_1=\left.\frac{(z-1)(z^2-4z+5)}{(z-1)(z-2)(z-3)}\right|_{z=1}=\frac{(1-4+5)}{(1-2)(1-3)}=\frac{2}{2}=1$$

$$\therefore A_2=\left.\frac{(z-2)(z^2-4z+5)}{(z-1)(z-2)(z-3)}\right|_{z=2}=\frac{(4-8+5)}{(2-1)(2-3)}=\frac{1}{-1}=-1$$

$$\therefore A_3=\left.\frac{(z-3)(z^2-4z+5)}{(z-1)(z-2)(z-3)}\right|_{z=3}=\frac{(9-12+5)}{(3-1)(3-2)}=\frac{2}{2}=1$$

$$\therefore \frac{X(z)}{z}=\frac{1}{(z-1)}+\frac{-1}{(z-2)}+\frac{1}{(z-3)} \qquad \mathit{and} \quad X(z)=\frac{z}{(z-1)}-\frac{z}{(z-2)}+\frac{z}{(z-3)}$$

- Now by method of inspection let us obtain the inverse transform for each of possible ROC.

Inverse Z-Transform by Partial Fraction Expansion

Example 2.: contd..

- Case1:|Z|<|1| each term corresponds to the left handed sequence

Recall here that

$$-b^n u(-n-1) \xleftrightarrow{ZT} = \frac{z}{z-b} \quad \text{ROC is} |z| < |b|$$

$$\frac{z}{(z-1)} \xleftrightarrow{IZT} -u(-n-1)$$

$$\frac{z}{(z-2)} \xleftrightarrow{IZT} -2^n u(-n-1) \quad and \quad \frac{z}{(z-3)} \xleftrightarrow{IZT} -3^n u(-n-1)$$

$$\therefore X(z) = \frac{z}{(z-1)} - \frac{z}{(z-2)} + \frac{z}{(z-3)} \xleftrightarrow{IZT} x(n) = -u(-n-1) + 2^n u(-n-1) - 3^n u(-n-1)$$

- Case2:|Z|>|3| each term corresponds to the right handed sequence

Recall here that

$$a^n u(n) \xleftrightarrow{ZT} = \frac{z}{z-a} \quad \text{ROC is} |z| > |a|$$

$$\frac{z}{(z-1)} \xleftrightarrow{IZT} u(n)$$

$$\frac{z}{(z-2)} \xleftrightarrow{IZT} 2^n u(n) \quad and \quad \frac{z}{(z-3)} \xleftrightarrow{IZT} 3^n u(n)$$

$$\therefore X(z) = \frac{z}{(z-1)} - \frac{z}{(z-2)} + \frac{z}{(z-3)} \xleftrightarrow{IZT} x(n) = u(n) - 2^n u(n) + 3^n u(n)$$

Inverse Z-Transform by Partial Fraction Expansion

Example 2.: contd..

- Case3:|2|<|Z|<|3| the term related to pole at 1and 2 corresponds to the right handed sequence whereas pole at 3 corresponds to left handed sequence

$$\frac{z}{(z-1)} \xleftrightarrow{IZT} u(n) \quad \frac{z}{(z-2)} \xleftrightarrow{IZT} 2^n u(n) \quad and \quad \frac{z}{(z-3)} \xleftrightarrow{IZT} -3^n u(-n-1)$$

$$\therefore X(z) = \frac{z}{(z-1)} - \frac{z}{(z-2)} + \frac{z}{(z-3)} \xleftrightarrow{IZT} x(n) = u(n) - 2^n u(n) - 3^n u(-n-1)$$

- Case4 :|1|<|Z|<|2| the term related to pole at 1corresponds to the right handed sequence whereas pole at 3 and 2 corresponds to left handed sequence

$$\frac{z}{(z-1)} \xleftrightarrow{IZT} u(n) \quad \frac{z}{(z-2)} \xleftrightarrow{IZT} -2^n u(-n-1) \quad and \quad \frac{z}{(z-3)} \xleftrightarrow{IZT} -3^n u(-n-1)$$

$$\therefore X(z) = \frac{z}{(z-1)} - \frac{z}{(z-2)} + \frac{z}{(z-3)} \xleftrightarrow{IZT} x(n) = u(n) + 2^n u(-n-1) - 3^n u(-n-1)$$

- For any other possible case X(z) does not exit.

Inverse Z-Transform by Partial Fraction Expansion

Example :3

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{\left(1+z^{-1}\right)^{2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \qquad |z|>1$$

- Long division to obtain B_o

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$\begin{array}{r|l} & \hfill 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & z^{-2}-3z^{-1}+2 \\ \cline{2-2} & \quad 5z^{-1}-1 \end{array}$$

- Let

$$X_0(z)=\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}=\frac{5z-z^{2}}{\left(z-\frac{1}{2}\right)\left(z-1\right)}$$

$$\therefore \frac{X_0(z)}{z}=\frac{5-z}{\left(z-\frac{1}{2}\right)\left(z-1\right)}=\frac{A_1}{z-\frac{1}{2}}+\frac{A_2}{z-1} \qquad \therefore X(z)=2+X_0(z)$$

Inverse Z-Transform by Partial Fraction Expansion

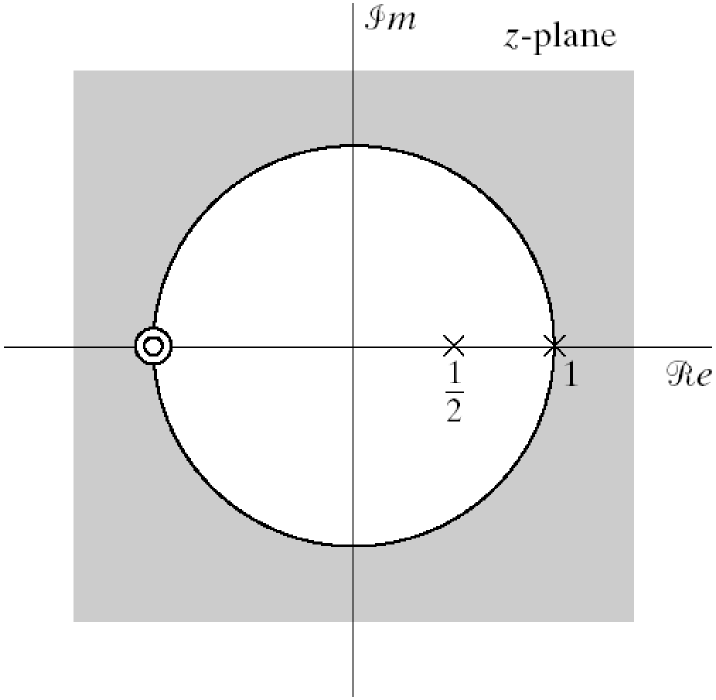
Example :3 Continued

$$A_1 = \left(z-\tfrac{1}{2}\right)\frac{(5-z)}{\left(z-\tfrac{1}{2}\right)(z-1)}\Bigg|_{z=\frac{1}{2}} = \frac{5-\frac{1}{2}}{\frac{1}{2}-1} = -9$$

$$\therefore X(z) = 2 + X_0(z)$$

$$A_2 = (z-1)\frac{(5-z)}{\left(z-\tfrac{1}{2}\right)(z-1)}\Bigg|_{z=1} = \frac{5-1}{1-\frac{1}{2}} = 8$$

$$\therefore X(z) = 2 - \frac{9z}{z-\dfrac{1}{2}} + \frac{8z}{z-1} \qquad |z| > 1$$



- ROC extends to infinity
  - Indicates right-sides sequence

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n]\text{-}8\text{u}[\text{n}]$$

Inverse Z-Transform by Power Series Method

- The z-transform is power series

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- In expanded form

$$X(z)=\cdots+x[-2]z^{2}+x[-1]z^{1}+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

- Z-transforms of this form can generally be inversed easily
- Especially useful for finite-length series
- Example1:

$$X(z)=z^{2}\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$$

$$=z^{2}\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-2}\right)$$

$$=z^{2}-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$$

$$\therefore x[n]=\begin{cases} 1 & n=-2 \\ -\dfrac{1}{2} & n=-1 \\ -1 & n=0 \\ \dfrac{1}{2} & n=1 \\ 0 & n=2 \end{cases}$$

$$\text{also}\ \ x[n]=\delta[n+2]-\frac{1}{2}\delta[n+1]-\delta[n]+\frac{1}{2}\delta[n-1]$$

Inverse Z-Transform by Power Series Method

- Example2:

$$\boldsymbol{X}(\boldsymbol{z}) = 4\boldsymbol{z}^2 + 2 + 3\boldsymbol{z}^{-1} \qquad 0 < |z| < \infty$$

$$x[n] = \begin{cases} 4, & \mathrm{n} = \text{-}2 \\ 2, & \mathrm{n} = 0 \\ 3, & \mathrm{n} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

- Example3: $X(z) = \dfrac{1}{1 - az^{-1}} \qquad |\boldsymbol{z}| > |\boldsymbol{a}|$

- Roc indicate the right handed sequence. So X(z) is a power series with –ve powers of z. To obtain the power series in –ve powers of z we have to perform the long division

Inverse Z-Transform by Power Series Method

Example 3 Contd…

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots = X(z) \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad + a^2z^{-2} \\
 & \qquad \underline{+ a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad + a^3z^{-3} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$x[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$x[n] = a^n u[n]$$

Inverse Z-Transform by Power Series Method

Example 4 Consider now $X(z)=\dfrac{1}{1-az^{-1}}$ $\quad |z|<|a|$

$$\underline{-a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3}} - \cdots = X(z)$$

$$-az^{-1}+1 \;\big)\; 1$$

$$\underline{1-a^{-1}z}$$

$$a^{-1}z$$

$$\underline{a^{-1}z-a^{-2}z^{2}}$$

$$+a^{-2}z^{2}$$

$$\underline{+a^{-2}z^{2}-a^{-3}z^{3}}$$

$$+a^{-3}z^{3}$$

$$\vdots$$

$$x[n]=\begin{cases} -a^{n} & n\le -1 \\ 0 & n>0 \end{cases}$$

$$x[n]=-a^{n}u[-n-1]$$

Inverse Z-Transform by Power Series Method

Example 4

$$X(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2} z^{-1}} \qquad |z| > \left|\tfrac{1}{2}\right|$$

$$2 + 2z^{-1} + z^{-2} + \tfrac{1}{2} z^{-3} + \left(\tfrac{1}{2}\right)^2 z^{-4} + \cdots = X(z)$$

$$1 - \tfrac{1}{2} z^{-1} \;\big)\; 2 + z^{-1}$$

$$\begin{array}{l}
\underline{2 - z^{-1}} \\
2z^{-1} \\
\underline{2z^{-1} - z^{-2}} \\
+ z^{-2} \\
\underline{z^{-2} - \left(\tfrac{1}{2}\right) z^{-3}} \\
+\left(\tfrac{1}{2}\right) z^{-3} \\
\left(\tfrac{1}{2}\right) z^{-3} - \left(\tfrac{1}{2}\right)^2 z^{-4} \\
\vdots
\end{array}$$

$$x[n] = \begin{cases} 2 & \text{n} = 0 \\ \left(\tfrac{1}{2}\right)^{n-2} & \text{n} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore x[n] = 2\delta(n) + \left(\tfrac{1}{2}\right)^{n-2} u[n-1]$$

Inverse Z-Transform by Power Series Method

Example 6

Consider the z-transform $X(z) = \ln\left(1 + az^{-1}\right),\ |z| > |a|$

$$|z| > |a| \Rightarrow \left|az^{-1}\right| < 1$$

$$\text{if} \quad -1 < x \le 1 \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

$$\therefore X(z) = \ln\left(1 + az^{-1}\right) = az^{-1} - \frac{a^2z^{-2}}{2} + \frac{a^3z^{-3}}{3} - \cdots + (-1)^{n+1}\frac{a^nz^{-n}}{n} + \cdots$$

$$\therefore X(z) = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{a^nz^{-n}}{n} \qquad |z| > |a|$$

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n} & n \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = \frac{-(-a)^n}{n}u[n-1]$$

Inverse Z-Transform by Power Series Method

Example 6

Consider the z-transform $X(z)=e^{z^2}$, ROC is all z except $|z|=\infty$

Using power series $e^{a}=\sum_{k=0}^{\infty}\frac{a^{k}}{k!}$

$$\therefore e^{z^2}=\sum_{k=0}^{\infty}\frac{\left(z^{2}\right)^{k}}{k!}$$

$$\therefore X[z]=\sum_{k=0}^{\infty}\frac{\left(z^{2k}\right)}{k!}=1+\frac{Z^{2}}{1!}+\frac{Z^{4}}{2!}+\frac{Z^{6}}{3!}$$

X(z) is power series of +ve powers of z . So x(n) is LHS.

$$\therefore x(n)=\begin{cases}0 & \text{n} > 0 \text{ and for odd n}\\ \dfrac{1}{\left(\frac{-n}{2}\right)!} & \text{otherwise}\end{cases}$$