The z-Transform: Transform analysis

Quote of the Day

Nothing in education is so astonishing as the amount of ignorance it accumulates in the form of inert facts.

- Henry Adams

Difference equations

- The input output relation of discrete time system characterized by difference equation.

$$\sum_{i=0}^{N} a_i y(n-i) = \sum_{i=0}^{M} b_i x(n-i) \qquad \because \text{Assume } a_0 = 1$$

$$y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{i=1}^{N} a_i y(n-i)$$

$$y(n) = \sum_{i=0}^{M} b_i x(n-i) - 1\sum_{i=1}^{N} a_i y(n-i) \xleftrightarrow{ztransform} Y(z) = \sum_{i=0}^{M} b_i z^{-i} X(z) - \sum_{i=1}^{N} a_i z^{-i} Y(z)$$

$$\therefore \frac{Y(z)}{X(z)} = H(Z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

- Where H(z) is the z-transform of impulse response h(n) also referred as transfer function because it represents the ratio of input and output function.

Stability and Causality

- Consider a system with impulse response h[n]

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k)$$

$$y(n)=\cdots h(-2)x(n+2)+h(-1)x(n+1)+h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)\cdots$$

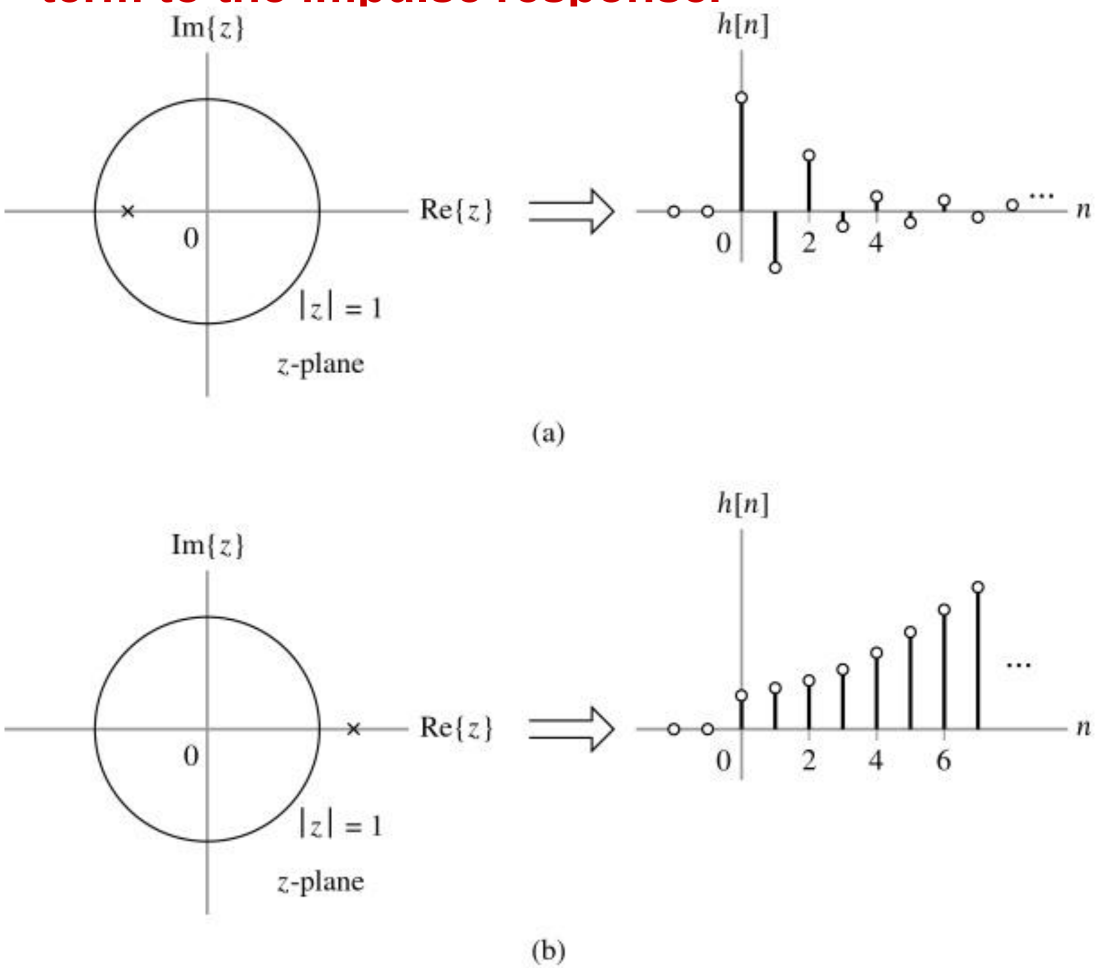
- Future instant +Present + Past instants of I/P
- For a system to be causal only present and past I/P terms should exist in the above expression.
- Therefore for system to be causal h(n) should be zero for –ve instant of time.
- If system is causal then it is required that the h(n) should be RHS and ROC must be |z|>|r|
- Now for stable system

$$\left|y(n)\right|=\left|\sum_{k=-\infty}^{\infty}h(k)x(n-k)\right|=\sum_{k=-\infty}^{\infty}\left|h(k)\right|\left|x(n-k)\right|<\infty$$

- If |x(n-k)| is Mx then for system to be stable it is required that $\sum_{k=-\infty}^{\infty}\left|h(k)\right|<\infty$. i.e if system is stable then it is required that the h(n) should be absolutely summable.

Figure 7.14 (p. 583)

The relationship between the location of a pole and the impulse response characteristics for a causal system. (a) A pole inside the unit circle contributes an exponentially decaying term to the impulse response. (b) A pole outside the unit circle contributes an exponentially increasing term to the impulse response.

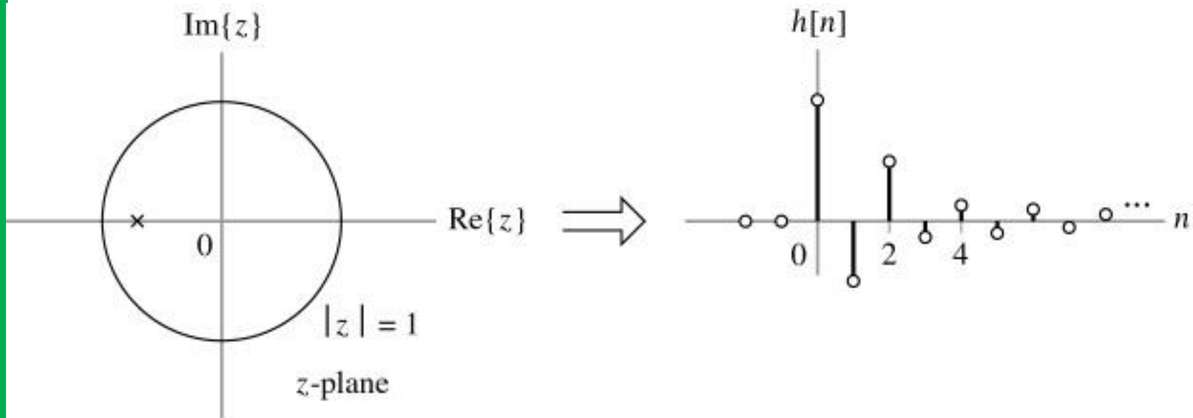


- If h(n) is R.H.S & |r| <1 then Causal, stable system

- If h(n) is R.H.S & |r| >1 Then system may be Causal but cannot be stable

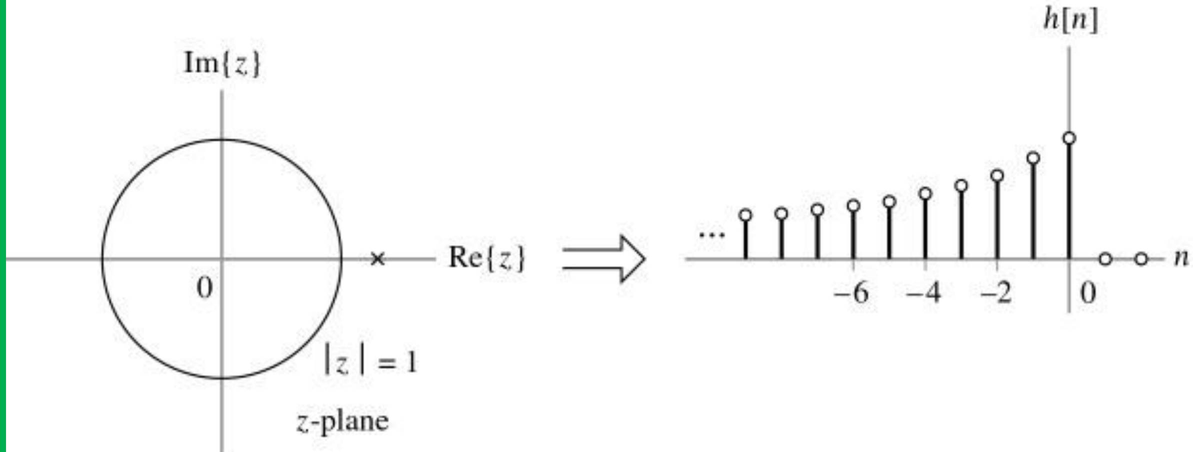
Figure 7.15 (p. 583)

The relationship between the location of a pole and the impulse response characteristics for a stable system. (a) A pole inside the unit circle contributes a right-sided term to the impulse response. (b) A pole outside the unit circle contributes a left-sided term to the impulse response.



(a)

- If h(n) is R.H.S & |r| <1 then Causal, stable system

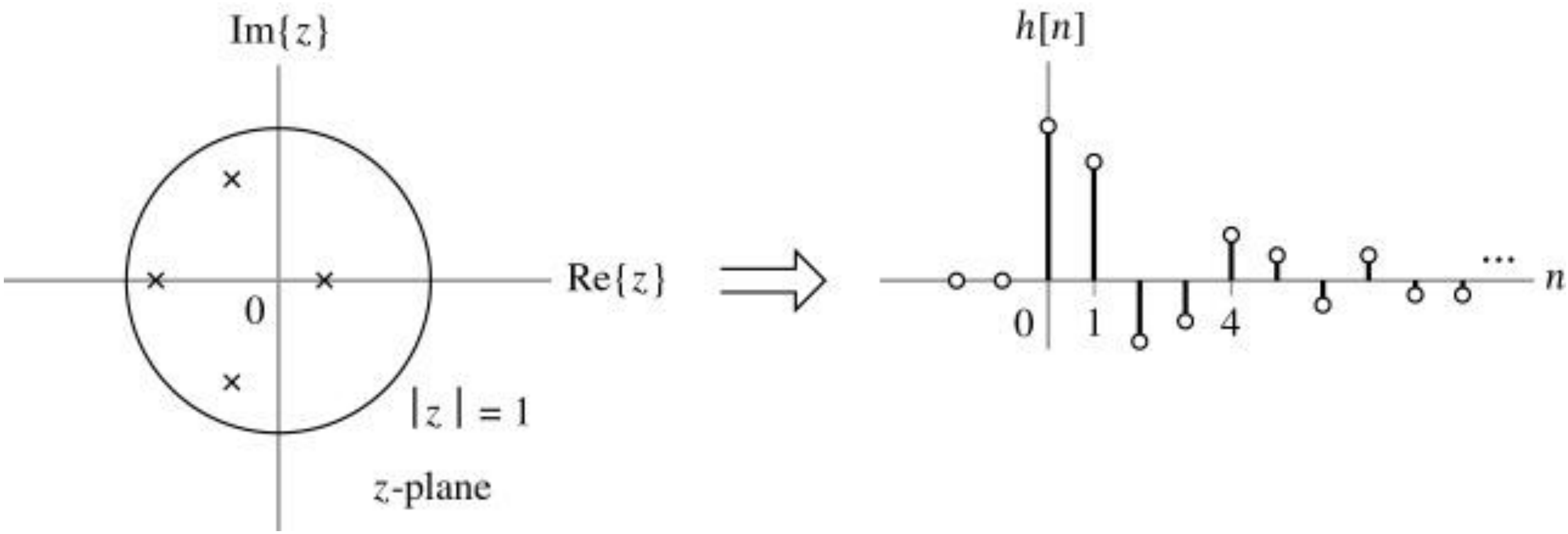


(b)

- If h(n) is L.H.S & |r| >1 then Non causal, stable system

Figure 7.16 (p. 584)

A system that is both stable and causal must have all its poles inside the unit circle in the *z*-plane, as illustrated here.



In any other case except described until system can not be both stable and causal.(If h(n) is LHS and |r|<1)

Ex:1.A causal system has input x(n) and output y(n), find impulse response of the system,

$$x[n]=\delta[n]+\frac{1}{4}\delta[n-1]-\frac{1}{8}\delta[n-2] \qquad y[n]=\delta[n]-\frac{3}{4}\delta[n-1]$$

impulse response: $H(z)=\dfrac{Y(z)}{X(z)}$ $\qquad H(z)\xleftrightarrow{IZT}h(n)$

Obtain Y(z) and X(z) then take ratio of them to find H(z)

$$\therefore X(z)=1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2} \qquad \therefore Y(z)=1-\frac{3}{4}z^{-1}$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-\frac{3}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{z^2-\frac{3}{4}z}{z^2+\frac{1}{4}z-\frac{1}{8}}=\frac{z^2-\frac{3}{4}z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}$$

$$\therefore \frac{H(z)}{z}=\frac{z-\frac{3}{4}}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}=\frac{A_1}{\left(z-\frac{1}{4}\right)}+\frac{A_2}{\left(z+\frac{1}{2}\right)}$$

Ex:1, A causal system has input x(n) and output y(n), find impulse response of the system,

$$\therefore A_1 = \left.\frac{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}\right|_{z=\frac{1}{4}} = \frac{\frac{1}{4}-\frac{3}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{-2}{3} \qquad \therefore A_2 = \left.\frac{\left(z+\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}\right|_{z=-\frac{1}{2}} = \frac{-\frac{1}{2}-\frac{3}{4}}{-\frac{1}{2}-\frac{1}{4}} = \frac{5}{3}$$

$$\therefore H(z) = \frac{-z\,{}^{2}\!/\!_{3}}{\left(z-\frac{1}{4}\right)} + \frac{z\,{}^{5}\!/\!_{3}}{\left(z+\frac{1}{2}\right)}$$

Impulse response for causal system should be RHS

$$\therefore h(n) = {}^{-2}\!/\!_{3}\left({}^{1}\!/\!_{4}\right)^{n} u(n) + {}^{5}\!/\!_{3}\left({}^{-1}\!/\!_{2}\right)^{n} u(n)$$

$$\therefore h(n) = {}^{1}\!/\!_{3}\left[5\left({}^{-1}\!/\!_{2}\right)^{n} - 2\left({}^{1}\!/\!_{4}\right)^{n}\right] u(n)$$

Ex:2, A DT LTI system is given by

$$\therefore H(z)=\frac{3-4z^{-1}}{\left(1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}\right)}$$

Specify the ROC of H(z) and determine impulse response for the following conditions

- Stable system
- Causal system
- Non causal system

$$\therefore H(z)=\frac{3z^2-4z}{\left(z^2-\frac{7}{2}z+\frac{3}{2}\right)} \qquad \therefore \frac{H(z)}{z}=\frac{3z-4}{(z-3)(z-\frac{1}{2})}=\frac{A_1}{(z-3)}+\frac{A_2}{(z-\frac{1}{2})}$$

$$\therefore A_1=\left.\frac{3z-4}{(z-\frac{1}{2})}\right|_{z=3}=\frac{9-4}{3-\frac{1}{2}}=2 \qquad \therefore A_2=\left.\frac{3z-4}{(z-3)}\right|_{z=\frac{1}{2}}=\frac{\frac{3}{2}-4}{\frac{1}{2}-3}=1$$

$$\therefore H(z)=\frac{2z}{(z-3)}+\frac{z}{(z-\frac{1}{2})}$$

- Stable system: pole at 3 corresponds to LHS and pole at ½ corresponds to RHS ROC will be |1/2|<|z|<|3|

$$\therefore h(n)=-2(3)^n u(-n-1)+\left(\tfrac{1}{2}\right)^n u(n)$$

Ex:2, contd..

- Causal system: pole at 3 and ½ both corresponds to RHS ROC will be |z|>|3|

$$\therefore h(n) = 2(3)^n u(n) + \left(\tfrac{1}{2}\right)^n u(n)$$

- Non Causal system: pole at 3 and ½ both corresponds to LHS ROC will be |z|<|½|

$$\therefore h(n) = -2(3)^n u(-n-1) - \left(\tfrac{1}{2}\right)^n u(-n-1)$$

Ex:3, Step response of a LTI system is found to be y(n)=2(1/3)^{n}u[n]. Find out impulse response of the system.

$$H(z) = \frac{Y(z)}{X(z)} \qquad X(z) = \frac{z}{z-1} \qquad y(n) = 2\left(\tfrac{1}{3}\right)^n u(n) \xleftrightarrow{ZT} Y(z) = 2\frac{z}{z-1/3}$$

$$\therefore H(z) = \frac{2z/\left(z-\frac{1}{3}\right)}{z/z-1} = \frac{2(z-1)}{z-\frac{1}{3}} = \frac{2z}{z-\frac{1}{3}} - \frac{2}{z-\frac{1}{3}} = \frac{2z}{z-\frac{1}{3}} - z^{-1}\frac{2z}{z-\frac{1}{3}}$$

$$\therefore h(n) = 2\left[\left(1/3\right)^n u(n) - \left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

Ex:4, Find the impulse response if a stable and causal system described by difference equation

$$y(n)+\tfrac{1}{4}y(n-1)-\tfrac{1}{8}y(n-2)=2x(n)+\left(\tfrac{5}{4}\right)x(n-1)$$

Taking the ZT of the given equation.

$$Y(z)+\tfrac{1}{4}z^{-1}Y(z)-\tfrac{1}{8}z^{-2}Y(z)=2X(z)+\left(\tfrac{5}{4}\right)z^{-1}X(z)$$

$$Y(z)\left(1+\tfrac{1}{4}z^{-1}-\tfrac{1}{8}z^{-2}\right)=\left(2+\left(\tfrac{5}{4}\right)z^{-1}\right)X(z)$$

$$\therefore H(z)=\frac{Y(z)}{X(z)}=\frac{\left(2+\left(\tfrac{5}{4}\right)z^{-1}\right)}{\left(1+\tfrac{1}{4}z^{-1}-\tfrac{1}{8}z^{-2}\right)}=\frac{2z^2+\tfrac{5}{4}z}{z^2+\tfrac{1}{4}z-\tfrac{1}{8}}$$

$$\therefore \frac{H(z)}{z}=\frac{2z+\tfrac{5}{4}}{z^2+\tfrac{1}{4}z-\tfrac{1}{8}}=\frac{2z+\tfrac{5}{4}}{\left(z+\tfrac{1}{2}\right)\left(z-\tfrac{1}{4}\right)}=\frac{A_1}{\left(z+\tfrac{1}{2}\right)}+\frac{A_2}{\left(z-\tfrac{1}{4}\right)}$$

$$A_1=-\tfrac{1}{3}\quad\text{and}\quad A_2=\frac{7}{3}\qquad\qquad \therefore H(z)=-\frac{\tfrac{1}{3}z}{\left(z+\tfrac{1}{2}\right)}+\frac{\tfrac{7}{3}z}{\left(z-\tfrac{1}{4}\right)}$$

$$\therefore h(n)=-\tfrac{1}{3}\left(\tfrac{-1}{2}\right)^n u(n)+\tfrac{7}{3}\left(\tfrac{1}{4}\right)^n u(n)$$

Ex:5, Find the impulse function H(z) and unit sample response h(n) of the system described by difference equation. Also find stability of system.(July15,

$$y(n)-2y(n-1)+2y(n-2)=x(n)+\left(\tfrac{1}{2}\right)x(n-1)$$

Taking the ZT of the given equation.

$$Y(z)-2z^{-1}Y(z)+2z^{-2}Y(z)=X(z)+\left(\tfrac{1}{2}\right)z^{-1}X(z)$$

$$Y(z)\left(1-2z^{-1}+2z^{-2}\right)=\left(1+\left(\tfrac{1}{2}\right)z^{-1}\right)X(z)$$

$$\therefore H(z)=\frac{Y(z)}{X(z)}=\frac{\left(1+\left(\tfrac{1}{2}\right)z^{-1}\right)}{\left(1-2z^{-1}+2z^{-2}\right)}=\frac{z^2+\tfrac{1}{2}z}{z^2-2z+2}$$

$$\therefore \frac{H(z)}{z}=\frac{z+\tfrac{1}{2}}{(z-1+j)(z-1-j)}=\frac{A_1}{(z-1+j)}+\frac{A_2}{(z-1-j)}$$

$$A_1=\tfrac{1}{2}+\tfrac{3}{4}\mathrm{j}\quad\text{and}\quad A_2=\tfrac{1}{2}-\tfrac{3}{4}\mathrm{j}\qquad \therefore H(z)=-\frac{(\tfrac{1}{2}+\tfrac{3}{4}j)z}{\left(z-1.414e^{j3\pi/4}\right)}+\frac{(\tfrac{1}{2}-\tfrac{3}{4}j)z}{\left(z-1.414e^{-j3\pi/4}\right)}$$

The roots are not lying inside the unit circle hence not a stable system.

$$\text{and } h(n)=0.9014e^{j56.31}\left(1.414e^{j3\pi/4}\right)^n u(n)+0.9014e^{-j56.31}\left(1.414e^{-j3\pi/4}\right)^n u(n)$$

$$\therefore h(n) = 0.9014(1.414)^n \left(e^{j56.31} e^{\frac{j3\pi n}{4}} + e^{-j56.31} e^{\frac{-j3\pi n}{4}} \right) u(n)$$

$$\therefore h(n) = 0.9014(1.414)^n \left(e^{\frac{j3\pi n}{4} + j56.31} + e^{\frac{-j3\pi n}{4} - j56.31} \right) u(n)$$

$$\therefore h(n) = 0.9014(1.414)^n \left(e^{j\left(\frac{3\pi n}{4} + 56.31\right)} + e^{-j\left(\frac{3\pi n}{4} + 56.31\right)} \right) u(n) = 1.8028(1.414)^n \left(\cos\frac{\pi n}{4} + 56.31 \right) u(n)$$

Ex:6, Find impulse response of the causal system

$$y(n) - y(n-1) = x(n) + x(n-1)$$

- Taking the ZT of the given equation.

$$Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z) \qquad \Rightarrow Y(z)\left(1 - z^{-1}\right) = \left(1 + z^{-1}\right)X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)} = \frac{z+1}{z-1}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-1} + \frac{1}{z-1} = \frac{z}{z-1} + z^{-1}\frac{z}{z-1}$$

$$\therefore h(n) = u(n) + u(n-1)$$

Ex:7, Find the impulse response if a stable and causal system described by difference equation also find the magnitude response at zero frequency.

$$y(n) = \tfrac{5}{6}\, y(n-1) - \tfrac{1}{6}\, y(n-2) + x(n) - 2x(n-1)$$

Taking the ZT of the given equation.

$$Y(z) - \tfrac{5}{6} z^{-1} Y(z) + \tfrac{1}{6} z^{-2} Y(z) = X(z) - 2z^{-1} X(z)$$

$$Y(z)\left(1 - \tfrac{5}{6} z^{-1} + \tfrac{1}{6} z^{-2}\right) = \left(1 - 2z^{-1}\right) X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1-2z^{-1}\right)}{\left(1 - \tfrac{5}{6} z^{-1} + \tfrac{1}{6} z^{-2}\right)} = \frac{z^2 - 2z}{z^2 - \tfrac{5}{6} z + \tfrac{1}{6}}$$

$$\therefore \frac{H(z)}{z} = \frac{z-2}{\left(z-\tfrac{1}{2}\right)\left(z-\tfrac{1}{3}\right)} = \frac{A_1}{\left(z-\tfrac{1}{2}\right)} + \frac{A_2}{\left(z-\tfrac{1}{3}\right)}$$

$$A_1 = -9 \text{ and } A_2 = 10 \quad \therefore H(z) = -\frac{9z}{\left(z-\tfrac{1}{2}\right)} + \frac{10z}{\left(z-\tfrac{1}{3}\right)}$$

$$\therefore h(n) = -9\left(\tfrac{1}{2}\right)^n u(n) + 10\left(\tfrac{1}{3}\right)^n u(n)$$

To find the frequency response substitute $z=e^{j\Omega}$ in H(z)

$$\therefore H(z) = \frac{z^2 - 2z}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{z^2 - 2z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \Rightarrow \therefore H(e^{j\Omega}) = \frac{e^{j2\Omega} - 2e^{j\Omega}}{\left(e^{j\Omega} - \frac{1}{2}\right)\left(e^{j\Omega} - \frac{1}{3}\right)}$$

$$\therefore \left|H(e^{j\Omega})\right| = \frac{\left|e^{j2\Omega} - 2e^{j\Omega}\right|}{\left|\left(e^{j\Omega} - \frac{1}{2}\right)\right| \cdot \left|\left(e^{j\Omega} - \frac{1}{3}\right)\right|} = \left.\frac{\left|e^{j2\Omega} - 2e^{j\Omega}\right|}{\left|\left(e^{j\Omega} - \frac{1}{2}\right)\right| \cdot \left|\left(e^{j\Omega} - \frac{1}{3}\right)\right|}\right|_{\Omega=0}$$

$$\therefore \left|H(e^{j\Omega})\right|_{\Omega=0} = \frac{|1-2|}{\left|\left(1 - \frac{1}{2}\right)\right| \cdot \left|\left(1 - \frac{1}{3}\right)\right|} = 3$$

- Observe the magnitude response of system is magnitude of numerator polynomial by the magnitude of denominator polynomial of $H(e^{j\Omega})$.
- Similarly the phase response of system is phase of numerator polynomial minus the phase of denominator polynomial of $H(e^{j\Omega})$.

Ex:8, Determine the impulse response system described by difference equation.

$$y(n) - 2y(n-1) + y(n-2) = x(n) + 3x(n-3)$$

Taking the ZT of the given equation.

$$Y(z) - 2z^{-1}Y(z) + z^{-2}Y(z) = X(z) - 3z^{-3}X(z)$$

$$Y(z)\left(1 - 2z^{-1} + z^{-2}\right) = \left(1 - 3z^{-3}\right)X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1-3z^{-3}\right)}{\left(1-2z^{-1}+z^{-2}\right)} = \frac{z^{2}-3z^{-1}}{\left(z^{2}-2z+1\right)} = \frac{z^{2}-3z^{-1}}{(z-1)^{2}}$$

$$\therefore H(z) = \frac{z^{2}}{(z-1)^{2}} - \frac{3z^{-1}}{(z-1)^{2}} \qquad = z\frac{z}{(z-1)^{2}} - 3z^{-2}\frac{z}{(z-1)^{2}}$$

$$\therefore h(n) = (n+1)u(n+1) - 3(n-2)u(n-2)$$

Find inverse z- transform of

$$X(z)=\frac{5z(z-1)}{(z^2-1.6z+0.8)}=\frac{5z(z-1)}{(z-0.8-j0.4)(z-0.8+j0.4)}$$

$$\therefore \frac{X(z)}{z}=\frac{5(z-1)}{(z-0.8-j0.4)(z-0.8+j0.4)}=\frac{A_1}{(z-0.8-j0.4)}+\frac{A_2}{(z-0.8+j0.4)}$$

$$A_1=2.5+1.25\text{j} \quad \text{and } A_2=2.5-1.25\text{j}$$

$$A_1=2.8e^{j26.56} \quad \text{and } A_2=2.8e^{-j26.56}$$

$$\therefore X(z)=\frac{2.8e^{j26.56}z}{(z+0.8-j0.4)}+\frac{2.8e^{-j26.56}z}{(z+0.8+j0.4)}$$

$$X(z)=\frac{2.8e^{j26.56}z}{(z+0.89e^{j26.56})}+\frac{2.8e^{-j26.56}z}{(z+0.89e^{-j26.56})}$$

$$\therefore x(n)=2.8e^{j56.26}\left(0.89e^{j26.56}\right)^n u(n)+2.8e^{-j56.26}\left(0.89e^{-j26.56}\right)^n u(n)$$

$$\therefore x(n)=2.8(0.89)^n\left(e^{j56.26(n+1)}+e^{-j56.26(n+1)}\right) \quad =2.8(0.89)^n\cos(56.26(n+1))$$

Find inverse z- transform of

$$X(z)=\frac{9}{(z+2)(z-0.5)^2}$$

$$\therefore \frac{X(z)}{z}=\frac{9}{z(z+2)(z-0.5)^2}=\frac{A_1}{z}+\frac{A_2}{(z+2)}+\frac{C_1}{(z-0.5)}+\frac{C_2}{(z-0.5)^2}$$

$$A_1=\frac{9}{(2)(-0.5)^2}=4.5 \quad \therefore A_2=\frac{9}{-2(-2-0.5)^2}=-0.72 \quad \therefore C_1=\frac{9}{0.5(0.5+2)}=7.2$$

$$\text{and } C_2=\frac{d}{dz}\frac{9}{(z^2+2z)}\bigg|_{z=0.5}=\frac{-9(2z+2)}{(z^2+2z)^2}\bigg|_{z=0.5}=\frac{-9(2\times0.5+2)}{(0.5^2+2\times0.5)^2}=-5.355$$

$$\therefore X(z)=4.5-0.72\frac{z}{(z+2)}+7.2\frac{z}{(z-0.5)}-5.355\frac{z}{(z-0.5)^2}$$

$$\therefore x(n)=4.5\delta(n)-0.72(2)^n u(n)+7.2(0.5)^n u(n)-5.355n(0.5)^{n-1}u(n)$$

Some examples for exercise from OP

1) $y(n)=\frac{5}{6}y(n-1)-\frac{1}{6}y(n-2)+x(n)-2x(n-1)$

Determine

i) System function H(z) and magnitude response at zero frequency.

ii) Impulse response of the system.

iii) Output y(n) for $x(n)=\delta(n)-\frac{1}{3}\delta(n-1)$

2) An LTI system is described by the equation

$$y(n)=x(n)+0.8\,x(n-1)+0.8x(n-2)-0.49y(n-2)$$

3) Determine the transfer function H(z) of the system and also sketch the poles and zeros.

Find the difference – equation description for a system with transform function :

$$H(z)=\frac{y(z)}{x(z)}=\frac{5z+2}{z^2+3z+2}.$$

4) A discrete LTI system is characterized by the difference equation,

$y(n)=y(n-1)+y(n-2)+x(n-1)$

Find the system function H(z) and indicate the ROC if the system, i) Stable ii) Causal.

Also determine the unit sample response of the stable system. **(10 Marks)**

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Examples for exercise from QP contd..

a. A system has impulse response $h(n)=\left(\frac{1}{2}\right)^n u(n)$, determine the input to the system if the output is given by,

$$y(n)=\frac{1}{3}u(n)+\frac{2}{3}\left(-\frac{1}{2}\right)^n u(n).$$

(08 Marks)

b. A systems has impulse response h (n) $(\frac{1}{3})^n$ u (n). Determine the transfer function. Also determine the input to the system if the output is given by:

$$y(n)=\frac{1}{2}u(n)+\frac{1}{4}\left(-\frac{1}{3}\right)^n u(n).$$

(05 Marks)

8 a. A system has the transfer function,

$$H(z)=\frac{2}{1-0.9e^{j\frac{\pi}{4}}z^{-1}}+\frac{2}{1-0.9e^{j\frac{\pi}{4}}z^{-1}}+\frac{3}{1+2z^{-1}}.$$

Find the impulse response assuming the system is (i) stable and (ii) causal. **(10 Marks)**

b. A system is described by the difference equation :

$$y(n)-y(n-1)+\frac{1}{4}y(n-2)=x(n)+\frac{1}{4}x(n-1)-\frac{1}{8}x(n-2).$$

Find the transfer function of the system. **(05 Marks)**