The z-Transform: Unilateral Z-Transform

Quote of the Day

“Without your involvement you can't succeed. With your involvement you can't fail”.

-DR. A.P. J Abdul Kalam

Unilateral Z-Transform

- The unilateral z-transform of x(n) is defined for only R.H.S. sequence. The mathematical expression is

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}$$

- We say $x(n)\xleftrightarrow{unilateralZT}X(z)$
- Unilateral transform of shifted sequence consider x(n-1)

$$UZ\left(x(n-1)\right)=\sum_{n=0}^{\infty}x(n-1)z^{-n}$$

$$\text{Put }(n-1)=m$$
$$\text{if } n=0, m=-1,$$
$$\text{if } n=\infty, m=\infty$$

$$=\sum_{m=-1}^{\infty}x(m)z^{-m-1}$$

$$=z^{-1}\left[\sum_{m=0}^{\infty}x(m)z^{-m}+x(-1)z\right]$$

$$=z^{-1}\sum_{m=0}^{\infty}x(m)z^{-m}+x(-1)$$

$$x(n-1)\xleftrightarrow{unilateralZT}z^{-1}X(z)+x(-1)$$

Unilateral Z-Transform

- Unilateral transform of shifted sequence consider x(n-2).

$$UZ\left(x(n-2)\right)=\sum_{n=0}^{\infty}x(n-2)z^{-n}$$

Put $\left(n-2\right)=m$

if $n=0, m=-2,$

if $n=\infty, m=\infty$

$$=\sum_{m=-2}^{\infty}x(m)z^{-m-2}$$

$$=z^{-2}\left[\sum_{m=0}^{\infty}x(m)z^{-m}+x(-1)z+x(-2)z^{2}\right]$$

$$=z^{-2}\sum_{m=0}^{\infty}x(m)z^{-m}+x(-1)z^{-1}+x(-2)$$

$$\therefore x(n-2)\xleftrightarrow{unilateralZT}z^{-2}X(z)+x(-1)z^{-1}+x(-2)$$

Similarly

$$\therefore x(n-k)\xleftrightarrow{unilateralZT}z^{-k}X(z)+x(-1)z^{-(k-1)}+x(-2)z^{-(k-2)}+\ldots+x(-k)$$

Unilateral Z-Transform

- Unilateral transform of shifted sequence consider x(n+1).

$$UZ\left(x(n+1)\right)=\sum_{n=0}^{\infty}x(n+1)z^{-n}$$

$$=\sum_{m=1}^{\infty}x(m)z^{-m+1}$$

$$=z\left[\sum_{m=0}^{\infty}x(m)z^{-m}-x(0)\right]$$

Put $(n+1)=m$
if $n=0, m=1,$
if $n=\infty, m=\infty$

$$\therefore x(n+1)\xleftrightarrow{unilateralZT}zX(z)-zx(0)$$

$$UZ\left(x(n+2)\right)=\sum_{n=0}^{\infty}x(n+2)z^{-n}=\sum_{m=2}^{\infty}x(m)z^{-m+2}$$

$$=z^2\left[\sum_{m=0}^{\infty}x(m)z^{-m}-x(0)-x(1)z^{-1}\right]$$

$$\therefore x(n+2)\xleftrightarrow{unilateralZT}z^2X(z)-z^2x(0)-zx(1)$$

$$\therefore x(n+k)\xleftrightarrow{unilateralZT}z^kX(z)-x(0)z^k-x(1)z^{(k-1)}-\ldots-x(k-1)z$$

Review of unilateral ZT

$$x(n-1) \xleftrightarrow{unilateralZT} z^{-1}X(z)+x(-1)$$

$$\therefore x(n-2) \xleftrightarrow{unilateralZT} z^{-2}X(z)+x(-1)z^{-1}+x(-2)$$

$$\therefore x(n-k) \xleftrightarrow{unilateralZT} z^{-k}X(z)+x(-1)z^{-(k-1)}+x(-2)z^{-(k-2)}+...+x(-k)$$

$$\therefore x(n+1) \xleftrightarrow{unilateralZT} zX(z)-zx(0)$$

$$\therefore x(n+2) \xleftrightarrow{unilateralZT} z^{2}X(z)-z^{2}x(0)-zx(1)$$

$$\therefore x(n+k) \xleftrightarrow{unilateralZT} z^{k}X(z)-x(0)z^{k}-x(1)z^{(k-1)}-...-x(k-1)z$$

- These Unilateral Z-transform pairs are mainly used to find the solution of difference equation.

Difference equations

- The input output relation of discrete time system characterized by difference equation.

$$\sum_{i=0}^{N} a_i y(n-i) = \sum_{i=0}^{M} b_i x(n-i)$$

- If $a_0=1$ $y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{i=1}^{N} a_i y(n-i)$

- Unilateral Z-transform pairs shall be used to find the solution of these kind of difference equation.
- Memorize natural response of the system is obtained by considering zero input with given initial condition.
- Memorize forced response of the system is obtained by considering zero initial condition and given input x[n].

Solution of Difference equations

- Example 1: Solve the difference equation

$$y(n+2)-\tfrac{3}{2}y(n+1)+\tfrac{1}{2}y(n)=\left(\tfrac{1}{4}\right)^n$$

with initial conditions y(0)=10, & y(1)=4. Use unilateral Z-transform.

⟹ Taking the unilateral ZT of the given equation.

$$\left\{z^2Y(z)-z^2y(0)-zy(1)\right\}-\tfrac{3}{2}\left\{zY(z)-zy(0)\right\}+\tfrac{1}{2}Y(z)=\frac{z}{z-\frac{1}{4}}$$

$$\therefore Y(z)\left\{z^2-\tfrac{3}{2}z+\tfrac{1}{2}\right\}-10z^2-4z+\tfrac{3}{2}\left\{10z\right\}=\frac{z}{z-\frac{1}{4}}$$

$$\therefore Y(z)\left\{z^2-\tfrac{3}{2}z+\tfrac{1}{2}\right\}=\frac{z}{z-\frac{1}{4}}+10z^2-11z$$

$$\therefore Y(z)=\frac{z}{\left(z-\frac{1}{4}\right)\left\{z^2-\frac{3}{2}z+\frac{1}{2}\right\}}+\frac{\left(10z^2-11z\right)}{\left\{z^2-\frac{3}{2}z+\frac{1}{2}\right\}}$$

$$\therefore \frac{Y(z)}{z}=\frac{1+(10z-11)\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{4}\right)(z-1)\left(z-\frac{1}{2}\right)}$$

Solution of Difference equations

- Example 1 contd..: Now use partial fraction expansion to obtain y(n).

$$\therefore \frac{Y(z)}{z} = \frac{1+(10z-11)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})} = \frac{A_1}{(z-\frac{1}{4})} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-\frac{1}{2})}$$

$$\therefore A_1 = (z-\tfrac{1}{4})\frac{1+(10z-11)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{4}} = \frac{1+(10\frac{1}{4}-11)(\frac{1}{4}-\frac{1}{4})}{(\frac{1}{4}-1)(\frac{1}{4}-\frac{1}{2})} = \frac{1}{(\frac{-3}{4})(\frac{-1}{4})} = \frac{16}{3}$$

$$\therefore A_2 = (z-1)\frac{1+(10z-11)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}\Bigg|_{z=1} = \frac{1+(10-11)(1-\frac{1}{4})}{(1-\frac{1}{4})(1-\frac{1}{2})} = \frac{\frac{1}{4}}{(\frac{3}{4})(\frac{1}{2})} = \frac{2}{3}$$

$$\therefore A_3 = (z-\tfrac{1}{2})\frac{1+(10z-11)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{2}} = \frac{1+(10\frac{1}{2}-11)(\frac{1}{2}-\frac{1}{4})}{(\frac{1}{2}-\frac{1}{4})(\frac{1}{2}-1)} = \frac{-\frac{1}{2}}{(\frac{1}{4})(\frac{-1}{2})} = 4$$

$$\therefore Y(z) = \frac{\frac{16}{3}z}{(z-\frac{1}{4})} + \frac{\frac{2}{3}z}{(z-1)} + \frac{4z}{(z-\frac{1}{2})}$$

$$\therefore Y(z) = \frac{\frac{16}{3}z}{(z-\frac{1}{4})} + \frac{\frac{2}{3}z}{(z-1)} + \frac{4z}{(z-\frac{1}{2})} \xleftrightarrow{ZT} y(n) = \tfrac{16}{3}\left(\tfrac{1}{4}\right)^n u(n) + \tfrac{2}{3}u(n) + 4\left(\tfrac{1}{2}\right)^n u(n)$$

Solution of Difference equations

- Example 2: Solve the difference equation with x(n)=δ(n) if $y(n)+y(n-2)=x(n)$

  with initial conditions y(-1)=1, & y(-2)=0.

Taking the unilateral ZT of the given equation.

$$Y(z)+\left\{z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right\}=X(z)$$

$$Y(z)+z^{-2}Y(z)+z^{-1}=1 \quad \therefore Y(z)\left(1+z^{-2}\right)=1-z^{-1}$$

$$\Rightarrow Y(z)=\frac{1-z^{-1}}{\left(1+z^{-2}\right)}=\frac{z^2-z}{z^2+1}\Rightarrow\frac{Y(z)}{z}=\frac{z-1}{(z+j)(z-j)}=\frac{A_1}{(z+j)}+\frac{A_2}{(z-j)}$$

$$A_1=\frac{-j-1}{-2j}=0.707e^{\frac{-j\pi}{4}} \quad A_2=\frac{j-1}{2j}=0.707e^{\frac{j\pi}{4}}$$

$$y(n)=0.707e^{\frac{-j\pi}{4}}(-j)^n u(n)+0.707e^{\frac{j\pi}{4}}(j)^n u(n)$$

$$\therefore y(n)=0.707\left[e^{\frac{-j\pi}{4}}\left(e^{\frac{-j\pi}{2}}\right)^n+e^{\frac{j\pi}{4}}\left(e^{\frac{j\pi}{2}}\right)^n\right]u(n)=1.414\cos\left(\tfrac{\pi n}{2}+\tfrac{\pi}{4}\right)$$

Solution of Difference equations

- Example 3: Solve the difference equation with $x(n)=(1/4)^n u(n)$ if $y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=2x(n)$ with initial conditions y(-1)=0, & y(-2)=0.

⟹ Taking the unilateral ZT of the given equation.

$$Y(z)-\tfrac{3}{4}\left\{z^{-1}Y(z)+y(-1)\right\}+\tfrac{1}{8}\left\{z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right\}=2X(z)$$

$$Y(z)-\tfrac{3}{4}z^{-1}Y(z)+\tfrac{1}{8}z^{-2}Y(z)=2\frac{z}{z-\frac{1}{4}}$$

$$Y(z)\left(1-\tfrac{3}{4}z^{-1}+\tfrac{1}{8}z^{-2}\right)=\frac{2}{\left(1-\frac{1}{4}z^{-1}\right)} \quad \Rightarrow Y(z)=\frac{2z}{\left(z-\frac{1}{4}\right)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)}$$

$$\Rightarrow \frac{Y(z)}{z}=\frac{2z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}=\frac{A_1}{z-\frac{1}{2}}+\frac{C_1}{z-\frac{1}{4}}+\frac{C_2}{\left(z-\frac{1}{4}\right)^2}$$

$$\therefore A_1=\left.\frac{2z^2}{\left(z-\frac{1}{4}\right)^2}\right|_{z=\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{16}}=8 \quad \therefore C_1=\left.\frac{2z^2}{\left(z-\frac{1}{2}\right)}\right|_{z=\frac{1}{4}}=\frac{\frac{1}{8}}{-\frac{1}{4}}=\frac{-1}{2}$$

Solution of Difference equations

Example 3 contd

$$\therefore C_2 = \frac{d}{dz}\left(\frac{2z^2}{z-\frac{1}{2}}\right)\Bigg|_{z=\frac{1}{4}} = \frac{\left(z-\frac{1}{2}\right)4z-2z^2}{\left(z-\frac{1}{2}\right)^2}\Bigg|_{z=\frac{1}{4}} = \frac{-\frac{1}{4}-\frac{1}{8}}{\frac{1}{16}} = -6$$

$$\Rightarrow Y(z) = \frac{8z}{z-\frac{1}{2}} - \frac{\frac{1}{2}z}{z-\frac{1}{4}} - \frac{6z}{\left(z-\frac{1}{4}\right)^2}$$

$$y(n) = 8\left(\tfrac{1}{2}\right)^n u(n) - \tfrac{1}{2}\left(\tfrac{1}{4}\right)^n u(n) - 6n\left(\tfrac{1}{4}\right)^{n-1} u(n)$$

Example 4: Solve the difference equation $y(n) = y(n-1) - y(n-2) + 2$; n≥0 with initial conditions: y(-1)=2, & y(-2)=1 to obtain forced and natural response

$$\Rightarrow Y(z) - \left\{z^{-1}Y(z) + y(-1)\right\} + \left\{z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right\} = \frac{2z}{z-1}$$

$$Y(z) = \frac{2z^3+z^2-2z}{(z-1)\left(1-z+z^2\right)} = \underbrace{\frac{2z^3}{(z-1)(z-e^{j\pi/3})(z-e^{-j\pi/3})}}_{\text{Forced response}} + \underbrace{\frac{z^2-2z}{(z-e^{j\pi/3})(z-e^{-j\pi/3})}}_{\text{Natural response}}$$

Solution of Difference equations

Example 4 contd

$$Y^n(Z)=\frac{z^2-2z}{(z-e^{j\pi/3})(z-e^{-j\pi/3})} \quad \therefore \frac{Y^n(Z)}{z}=\frac{z-2}{(z-e^{j\pi/3})(z-e^{-j\pi/3})}=\frac{A_1}{(z-e^{j\pi/3})}+\frac{A_2}{(z-e^{-j\pi/3})}$$

$$\therefore A_1=\left.\frac{z-2}{(z-e^{-j\pi/3})}\right|_{z=e^{j\pi/3}}=e^{j\pi/3} \quad \therefore A_2=\left.\frac{z-2}{(z-e^{j\pi/3})}\right|_{z=e^{-j\pi/3}}=e^{-j\pi/3}$$

$$\therefore Y^n(Z)=\frac{ze^{j\pi/3}}{(z-e^{j\pi/3})}+\frac{ze^{-j\pi/3}}{(z-e^{-j\pi/3})} \quad \therefore y^n(n)=e^{j\pi/3}e^{j\pi n/3}+e^{-j\pi/3}e^{-j\pi n/3}=2\cos\left(\frac{\pi(n+1)}{3}\right)$$

$$Y^f(Z)=\frac{2z^3}{(z-1)(z-e^{j\pi/3})(z-e^{-j\pi/3})}$$

$$\therefore \frac{Y^f(Z)}{z}=\frac{2z^2}{(z-1)(z-e^{j\pi/3})(z-e^{-j\pi/3})}=\frac{A_1}{(z-1)}+\frac{A_2}{(z-e^{j\pi/3})}+\frac{A_3}{(z-e^{-j\pi/3})}$$

$$\therefore A_1=\left.\frac{2z^2}{(z^2-z+1)}\right|_{z=\frac{1}{2}}=1 \quad \therefore A_2=\left.\frac{2z^2}{(z-1)(z-e^{-j\pi/3})}\right|_{z=e^{j\pi/3}}=\frac{2(-0.5+j\sqrt{3}/2)}{(-0.5+j\sqrt{3}/2)2j\sqrt{3}/2}=\frac{2}{j\sqrt{3}}$$

$$\therefore A_3=\left.\frac{2z^2}{(z-1)(z-e^{j\pi/3})}\right|_{z=e^{-j\pi/3}}==\frac{2(-0.5-j\sqrt{3}/2)}{(-0.5-j\sqrt{3}/2)2j\left(-\sqrt{3}/2\right)}=\frac{2}{-j\sqrt{3}}$$

Solution of Difference equations

Example 4 contd

$$\therefore Y^f(Z) = \frac{z}{(z-1)} + \frac{2z}{j\sqrt{3}(z-e^{j\pi/3})} - \frac{2z}{j\sqrt{3}(z-e^{-j\pi/3})}$$

$$\therefore y^f(n) = u(n) + \frac{2}{j\sqrt{3}} e^{j\pi n/3} - \frac{2}{j\sqrt{3}} e^{-j\pi n/3} = u(n) + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right) u(n)$$

$$\therefore y^c(n) = y^n(n) + y^f(n) = 2\cos\left(\frac{\pi(n+1)}{3}\right) u(n) + u(n) + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right) u(n)$$

Example 5: Solve the difference equation for initial conditions: y(-1)=2, & y(-2)=1 and x(n)=3u(n).

$$y(n) - \frac{1}{9} y(n-2) = x(n-1)$$

$$Y(z) - \frac{1}{9}\left\{z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right\} = \frac{3z^{-1}\cdot z}{z-1}$$

$$\therefore Y(z)\left(z^2 - \frac{1}{9}\right) - \frac{2}{9}z - \frac{1}{9}z^2 = \frac{3z^2}{z-1} \quad \therefore Y(z) = \frac{3z^2}{(z-1)\left(z^2 - \frac{1}{9}\right)} + \frac{z^2 + 2z}{9\left(z^2 - \frac{1}{9}\right)}$$

Solution of Difference equations

Example 5 contd

$$\therefore Y(z) = \frac{z^3 + 28z^2 - 2z}{9(z-1)(z-1/3)(z+1/3)}$$

$$\therefore \frac{Y(Z)}{z} = \frac{z^2 + 28z - 2}{9(z-1)(z-1/3)(z+1/3)} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-1/3)} + \frac{A_3}{(z+1/3)}$$

$$\therefore A_1 = \frac{1+28-2}{9(2/3)(4/3)} = \frac{9}{8} \qquad \therefore A_2 = \frac{1/9 + 28/3 - 2}{9(-2/3)(2/3)} = -\frac{67}{36}$$

$$\therefore A_3 = \frac{1/9 - 28/3 - 2}{9(-4/3)(-2/3)} = -\frac{101}{72}$$

$$\therefore Y(Z) = \frac{9z}{8(z-1)} - \frac{67z}{36(z-1/3)} - \frac{101z}{72(z+1/3)}$$

$$y(n) = \frac{9}{8}u(n) - \frac{67}{36}\left(\tfrac{1}{3}\right)^n u(n) - \frac{101}{72}\left(-\tfrac{1}{3}\right)^n u(n)$$

$$y(n) = \left(\frac{9}{8} - \frac{67}{36}\left(\tfrac{1}{3}\right)^n - \frac{101}{72}\left(-\tfrac{1}{3}\right)^n\right)u(n)$$

Solution of Difference equations

Example 6: Solve the difference equation using UZ for initial conditions: y(-1)=4, & y(-2)=10 and x(n)= $(1/4)^n$ u(n).

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n) \quad \text{for } n \geq 0.$$

$$\Rightarrow Y(z)-\tfrac{3}{2}\left\{z^{-1}Y(z)+y(-1)\right\}+\tfrac{1}{2}\left\{z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right\}=X(z)$$

$$Y(z)\left(1-\tfrac{3}{2}z^{-1}+\tfrac{1}{2}z^{-2}\right)-6+2z^{-1}+5=\frac{z}{z-\frac{1}{4}}$$

$$Y(z)\left(z^{2}-\tfrac{3}{2}z+\tfrac{1}{2}\right)=\frac{z^{3}}{(z-\frac{1}{4})}-2z+z^{2} \Rightarrow Y(z)=\frac{z^{3}+z(z-2)(z-\frac{1}{4})}{(z-\frac{1}{4})\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)}$$

$$\therefore Y(z)=\frac{z^{3}+(z-2)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})} \quad \therefore \frac{Y(z)}{z}=\frac{z^{2}+(z-2)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}$$

$$\therefore \frac{Y(z)}{z}=\frac{z^{2}+(z-2)(z-\frac{1}{4})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}=\frac{A_1}{(z-\frac{1}{4})}+\frac{A_2}{(z-1)}+\frac{A_3}{(z-\frac{1}{2})}$$

Solution of Difference equations

Example 6 contd

$$\therefore A_1 = \left.\frac{z^2+(z-2)(z-1/4)}{(z-1)(z-1/2)}\right|_{z=\frac{1}{4}} = \frac{\frac{1}{16}}{3/8} = \frac{1}{6}$$

$$\therefore A_2 = \left.\frac{z^2+(z-2)(z-1/4)}{(z-1/4)(z-1/2)}\right|_{z=1} = \frac{1-\frac{3}{4}}{\frac{3}{8}} = \frac{2}{3}$$

$$\therefore A_3 = \left.\frac{z^2+(z-2)(z-1/4)}{(z-1/4)(z-1)}\right|_{z=1/2} = \frac{1/4-3/16}{-1/8} = -\frac{1}{2}$$

$$\therefore Y(z) = \frac{1/6\, z}{(z-1/4)} + \frac{2/3\, Z}{(z-1)} - \frac{1/2\, z}{(z-1/2)}$$

$$y(n) = \tfrac{1}{6}\left(\tfrac{1}{4}\right)^n u(n) + \tfrac{2}{3}u(n) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u(n)$$

$$y(n) = \left\{\tfrac{1}{6}\left(\tfrac{1}{4}\right)^n + \tfrac{2}{3} - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n\right\}\cdot u(n)$$

Solution of Difference equations

Example 7: A linear shift invariant system is described by the difference equation

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=2x(n)$$

With y(-1)=1, y(-2)= -1. Find

i. The natural response of the system.
ii. The forced response of the system and
iii. The complete response of the system for a step input.

$$Y(z)-\tfrac{3}{4}\left\{z^{-1}Y(z)+y(-1)\right\}+\tfrac{1}{8}\left\{z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right\}=2X(z)$$

$$\therefore Y(z)\left(1-\tfrac{3}{4}z^{-1}+\tfrac{1}{8}z^{-2}\right)-\tfrac{3}{4}+\tfrac{1}{8}z^{-1}-\tfrac{1}{8}=\frac{2z}{z-1}$$

$$\therefore Y(z)\left(z^{2}-\tfrac{3}{4}z+\tfrac{1}{8}\right)=\frac{2z^{3}}{(z-1)}-\tfrac{1}{8}z+\tfrac{7}{8}z^{2}$$

$$\therefore \frac{Y^{f}(z)}{z}=\frac{2z^{2}}{(z-1)\left(z-\tfrac{1}{4}\right)\left(z-\tfrac{1}{2}\right)}$$

$$\therefore \frac{Y^{n}(z)}{z}=\frac{\tfrac{7}{8}z-\tfrac{1}{8}}{\left(z-\tfrac{1}{4}\right)\left(z-\tfrac{1}{2}\right)}$$

Solution of Difference equations

SDM JAIN MATT TRUST, JAIN AGMIT JAMKHANDI, ESTD:2014-15

Example 7 contd

$$\therefore \frac{Y^n(z)}{z} = \frac{\tfrac{7}{8}z - \tfrac{1}{8}}{(z-\tfrac{1}{4})(z-\tfrac{1}{2})} = \frac{A_1}{(z-\tfrac{1}{4})} + \frac{A_2}{(z-\tfrac{1}{2})}$$

$$\therefore A_1 = \frac{\tfrac{7}{32} - \tfrac{1}{8}}{(\tfrac{1}{4}-\tfrac{1}{2})} = -\tfrac{3}{8} \quad \therefore A_2 = \frac{\tfrac{7}{16} - \tfrac{1}{8}}{(\tfrac{1}{2}-\tfrac{1}{4})} = \tfrac{5}{4} \qquad \therefore Y^n(z) = \frac{-z\,\tfrac{3}{8}}{(z-\tfrac{1}{4})} + \frac{z\,\tfrac{5}{4}}{(z-\tfrac{1}{2})}$$

$$\boxed{\therefore y^n(n) = \left\{-\tfrac{3}{8}\left(\tfrac{1}{4}\right)^n + \tfrac{5}{4}\left(\tfrac{1}{2}\right)^n\right\}u(n)}$$

$$\text{Now}\quad \frac{Y^f(z)}{z} = \frac{2z^2}{(z-1)(z-\tfrac{1}{4})(z-\tfrac{1}{2})} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-\tfrac{1}{4})} + \frac{A_3}{(z-\tfrac{1}{2})}$$

$$\therefore A_1 = \frac{2}{(1-\tfrac{1}{4})(1-\tfrac{1}{2})} = \tfrac{16}{3} \quad \therefore A_2 = \frac{2\,\tfrac{1}{16}}{(\tfrac{1}{4}-1)(\tfrac{1}{4}-\tfrac{1}{2})} = \tfrac{2}{3} \quad \therefore A_3 = \frac{\tfrac{1}{2}}{(-\tfrac{1}{2})(\tfrac{1}{4})} = -4$$

$$\therefore Y^f(z) = \frac{\tfrac{16}{3}z}{(z-1)} + \frac{\tfrac{2}{3}z}{(z-\tfrac{1}{4})} - \frac{4z}{(z-\tfrac{1}{2})}$$

$$\boxed{\therefore y^f(n) = \left\{\frac{16}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^n - 4\left(\frac{1}{2}\right)\right\}^n u(n)}$$

$$\boxed{\therefore y^c(n) = y^n(n) + y^f(n) = \left\{\frac{16}{3} + \frac{7}{24}\left(\frac{1}{4}\right)^n - \frac{11}{4}\left(\frac{1}{2}\right)\right\}^n u(n)}$$

Solution of Difference equations

Example 8: A linear shift invariant system is described by the difference equation

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n)+x(n-1)$$

With y(-1)=1, y(-2)=-1 &. Find

i. The natural response of the system.
ii. The forced response of the system and
iii. The frequency response of the system for a step.

$$Y(z)-\tfrac{3}{4}\left\{z^{-1}Y(z)+y(-1)\right\}+\tfrac{1}{8}\left\{z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right\}=X(z)+z^{-1}X(z)$$

$$\Rightarrow\left(1-\tfrac{3}{4}z^{-1}+\tfrac{1}{8}z^{-2}\right)Y(z)-\tfrac{3}{4}+\tfrac{1}{8}z^{-1}-\tfrac{1}{8}=\frac{z+1}{z-1}\quad\Rightarrow Y(z)\left(1-\tfrac{3}{4}z^{-1}+\tfrac{1}{8}z^{-2}\right)=\frac{z+1}{z-1}+\frac{7}{8}-\frac{1}{8}z^{-1}$$

$$\Rightarrow Y(z)=\frac{z^3+z^2}{(z-1)(z-\frac{1}{4})(z-\frac{1}{2})}+\frac{\frac{7}{8}z^2-\frac{1}{8}z}{(z-\frac{1}{4})(z-\frac{1}{2})}$$

$$\therefore y^n(n)=\left\{-\tfrac{3}{8}\left(\tfrac{1}{4}\right)^n+\tfrac{5}{4}\left(\tfrac{1}{2}\right)^n\right\}u(n)$$

$$\therefore y^f(n)=\left\{\frac{16}{3}+\frac{5}{3}\left(\frac{1}{4}\right)^n-6\left(\frac{1}{2}\right)^n\right\}u(n)$$

$$\therefore y^c(n)=y^n(n)+y^f(n)=\left\{\frac{16}{3}+\frac{31}{24}\left(\frac{1}{4}\right)^n-\frac{19}{4}\left(\frac{1}{2}\right)^n\right\}u(n)$$

Find the UZ of the sequence $y(n)=x(n-2)$ where $x(n)=\alpha^n$

- $Y(z)=z^{-2}X(z)+z^{-1}x(-1)+x(-2)$
- $\therefore Y(z)=z^{-2}X(z)+z^{-1}\alpha^{-1}+\alpha^{-2}$
- Where $X(z)=z/(z-\alpha)$
- $\therefore Y(z)=z^{-1}\alpha^{-1}+\alpha^{-2}+z^{-1}/(z-\alpha)$
- $\therefore Y(z)=(\alpha^{-1}-z^{-1}+z\alpha^{-2}-\alpha^{-1}+z^{-1})/(z-\alpha)$
- $\therefore Y(z)=z\alpha^{-2}/(z-\alpha)$

Find the UZ of the sequence y(n)=x(n-2) where $x(n)=\{1,2,\underset{\uparrow}{3},4\}$ Find UZ of $z(n)=a^n u(n+1)$; a<1

- Unilateral ZT of x(n) $\Rightarrow X(z)=\sum_{n=0}^{1} x(n)z^{-n} = 3+4z^{-1}$

- UZT of y(n) $\Rightarrow \therefore x(n-2) \xleftrightarrow{unilateralZT} z^{-2}X(z)+x(-1)z^{-1}+x(-2)$

$$\therefore Y(z)=z^{-2}\{3+4z^{-1}\}+x(-1)z^{-1}+x(-2) = 3z^{-2}+4z^{-3}+2z^{-1}+1$$

- UZT of z(n) $\Rightarrow Z(z)=\sum_{n=0}^{\infty} a^n z^{-n} = \dfrac{z}{z-a}$ with ROC $|z|>|a|$

Problems for exercise

1. Solve the following difference equation using the unilateral z-transform.

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=x(n) \text{ for } n\geq 0$$

With initial conditions y(-1) = 2, y(-2) = 4 and $x(n)=\left(\frac{1}{5}\right)^n u(n)$.

2 b. Solve the following difference equation for the given initial conditions and input,

$$y(n)-\frac{1}{9}y(n-2)=x(n-1)$$

With $y(-1)=0$, $y(-2)=1$ and $x(n)=3u(n)$

3. Consider the system described the difference question $y(n) - 0.9y(n-1) = x(n)$. Find output if the input is $x(n) = u(n)$ and the initial condition on the output is $y(-1) = 2$. (08 Marks)

4. A linear shift invariant system is described by the difference equation.

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n)+x(n-1)$$

with $y(-1) = 0$ and $y(-2) = -1$.

Find:

i) The natural response of the system.

ii) The forced response of the system and

iii) The frequency response of the system for a step.